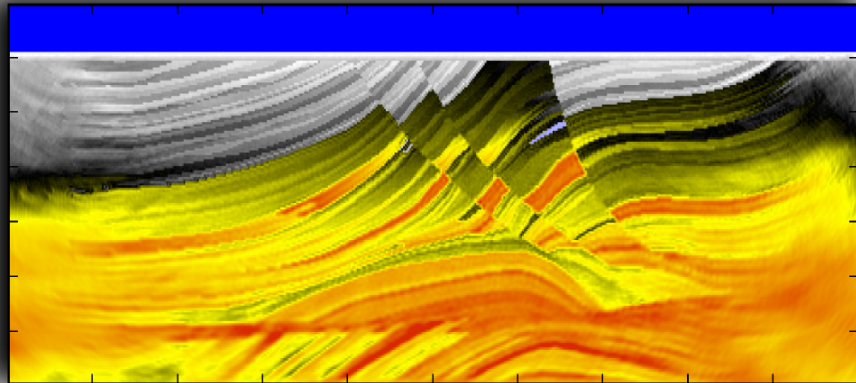
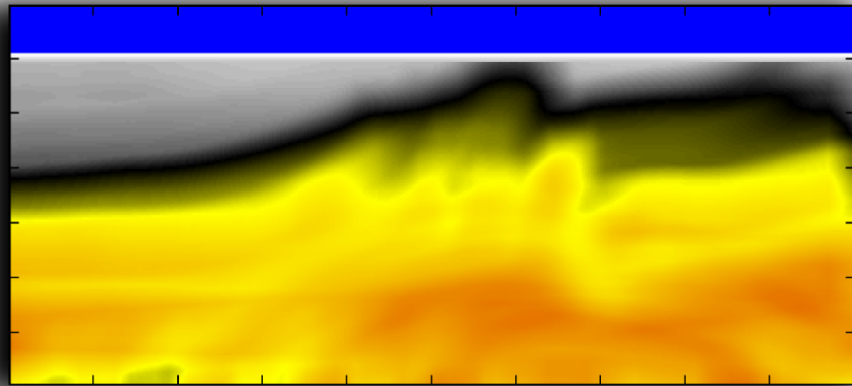


# SAVA

## User Manual



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Olaf Hellwig  
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Maik Linke



# **SAVA**

## **User Manual**

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Technische Universität Bergakademie Freiberg TUBAF (Germany)  
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## Authors

The DENISE code was first developed by Daniel Köhn, Olaf Hellwig and Denise De Nil at the Christian-Albrechts-Universität Kiel and TU Bergakademie Freiberg (Germany) from January to February 2012.

The anisotropic forward code is an extension of the 3D isotropic elastic FD code fd3D by Olaf Hellwig.

Different external libraries for timedomain filtering are used.

The copyright of the source codes are held by different persons:

cseife.c, cseife.h, lib\_stfinv, lib\_aff, lib\_fourier:

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This algorithm was part of seife.f. A current version of seife.f can be obtained from <http://www.software-for-seismometry.de/>

The Matlab implementation of a few SU routines, mainly used to read and write SU files in data pre-processing are:

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Author: M.D.Sacchi

Since then it has been developed and maintained by a development team: in alphabetical order,

Maik Linke (TU Bergakademie Freiberg),

(add other developers here in the future).

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# Chapter 1

## Introduction

The aim of Full Waveform Tomography (FWT) is to estimate the elastic material parameters in the underground. This can be achieved by minimizing the misfit energy between the modelled and field data using a gradient optimization approach. Because the FWT uses the full information content of each seismogram, structures below the seismic wavelength can be resolved. This is a tremendous improvement in resolution compared to traveltimes tomography (Pratt et al. [2002]).

The concept of full waveform tomography was originally developed by Albert Tarantola in the 1980s for the acoustic, isotropic elastic, and viscoelastic case (Tarantola [1984b,a, 1986, 1988]). First numerical implementations were realized at the end of the 1980s (Gauthier et al. [1986], Mora [1987], Pica et al. [1990]), but due to limited computational resources, the application was restricted to simple 2D synthetic test problems and small near offset datasets. At the beginning of the 1990s the original time domain formulation was transferred to a robust frequency domain approach (Pratt and Worthington [1990], Pratt [1990]). With the increasing performance of supercomputers moderately sized problems could be inverted with frequency domain approaches.

A spectacular result to prove the application of acoustic FWT on laboratory scale was presented by Pratt [1999] for ultrasonic tomography measurements on a simple block model. In a numerical blind test Brenders and Pratt [2007] achieved a very good agreement between their inversion result and the unknown true P-wave velocity model. The parallelization and performance optimizations of the frequency domain approach (see e.g. Sourbier et al. [2009a], Sourbier et al. [2009b]) lead to a wide range of acoustic FWT applications for problems on different scales, from the global scale, crustal scale over engineering and near surface scale, down to laboratory scale (Pratt [2004]).

Beside the application to geophysical problems, the acoustic FWT is also used to improve the resolution in medical cancer diagnostics (Pratt et al. [2007]). However, all these examples are restricted to the inversion of the acoustic material parameters: P-wave velocity, density and additionally the viscoacoustic damping  $Q_p$  for the P-waves. Even today the independent 2D FWT of all three isotropic elastic material parameters is still a challenge. Most elastic approaches invert for P-wave velocity only and use empirical relationships to deduce the distribution of S-wave velocity and density (Shipp and Singh [2002], Sheen et al. [2006]). Recently some authors also investigated the independent multiparameter FWT in the frequency domain (Choi et al. [2008a,b], Brossier [2009]).

In order to extract information about the structure and composition of the crust from seismic observations, it is necessary to be able to predict how seismic wavefields are affected by complex structures. Since exact analytical solutions to the wave equations do not exist for most subsurface configurations, the solutions can be obtained only by numerical methods. For iterative calculations of synthetic seismograms with limited computer resources fast and accurate modelling methods are needed.

The FD modelling/inversion program SAVA, is based on the FD approach described by Virieux [1986] and Levander [1988]. The present program SAVA has the following extensions

- considers propagation of seismic waves in general anisotropic elastic media
- is efficiently parallelized using domain decomposition with MPI,
- applies Convolutional Perfectly Matched Layer boundary conditions at the edges of the numerical mesh Komatitsch and Martin [2007].

In the following sections, we give an extensive description of the theoretical background, the different input parameters and show a few benchmark modelling and inversion applications.

## 1.1 Citation

If you use this code for your own research, please cite at least one article written by the developers of the package, for instance:

XX

or

(XX add more references here)

and/or other articles from (<http://www.geophysik.uni-kiel.de/~dkoehn/publications.htm>)

The corresponding Bib<sub>T</sub><sub>E</sub>X entries may be found in file `doc/USER_MANUAL/thesis.bib`.

## 1.2 Support

The development of the code was supported by the Christian-Albrechts-Universität Kiel, TU Bergakademie Freiberg, Deutsche Forschungsgemeinschaft (DFG), Bundesministerium für Bildung und Forschung (BMBF). The code was tested and optimized at the computing centres of Kiel University, TU Bergakademie Freiberg and the Hochleistungsrechenzentrum Nord (HLRN 1+2).

# Acknowledgments and contact

We thank for constructive discussions and further code improvements:

Wolfgang Rabbel (Christian-Albrechts-Universität Kiel).

Please e-mail your feedback, questions, comments, and suggestions to  
Daniel Köhn (`dkoehn-AT-geophysik.uni-kiel.de`).

## Chapter 2

# Theoretical Background

### 2.1 Equations of motion for an elastic medium

The propagation of waves in a general elastic medium can be described by a system of coupled linear partial differential equations. They consist of the equations of motion

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \quad (2.1)$$

which simply state that the momentum of the medium, the product of density  $\rho$  and the displacement velocity  $v_i$ , can be changed by surface forces, described by the stress tensor  $\sigma_{ij}$  or body forces  $f_i$ . These equations describe a general medium, like gas, fluid, solid or plasma. The material specific properties are introduced by additional equations which describe how the medium reacts when a certain force is applied. In the general anisotropic elastic case this can be described by a linear stress-strain relationship:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} \epsilon_{kl} + T_{ij} \\ \epsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{aligned} \quad (2.2)$$

where  $c_{ijkl}$  denotes the elastic tensor,  $\epsilon_{ij}$  the strain tensor,  $T_{ij}$  surface force source term and  $u_i$  the displacement vector. Using the definition of the particle velocity  $v_i = \frac{\partial u_i}{\partial t}$ , (2.1) and (2.2) can be transformed into a system of second order partial differential equations:

$$\begin{aligned} \rho \frac{\partial^2 u_i}{\partial t^2} &= \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \\ \sigma_{ij} &= c_{ijkl} \epsilon_{kl} + T_{ij} \\ \epsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{aligned} \quad (2.3)$$

This expression is called **Stress-Displacement** formulation. Another common form of the elastic equations of motion can be deduced by taking the time derivative of the stress-strain relationship and the strain tensor in Eq. (2.3). Since the elastic tensor  $c_{ijkl}$  does not depend on time, Eq. (2.3) can be written as:

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \quad (2.4)$$

$$\frac{\partial \sigma_{ij}}{\partial t} = c_{ijkl} \frac{\partial \epsilon_{kl}}{\partial t} + \frac{\partial T_{ij}}{\partial t} \quad (2.5)$$

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (2.6)$$

This expression is called **Stress-Velocity** formulation. For simple cases (2.3) and (2.4) can be solved analytically. More complex problems require numerical solutions. One possible approach for a numerical solution is described in the next section.

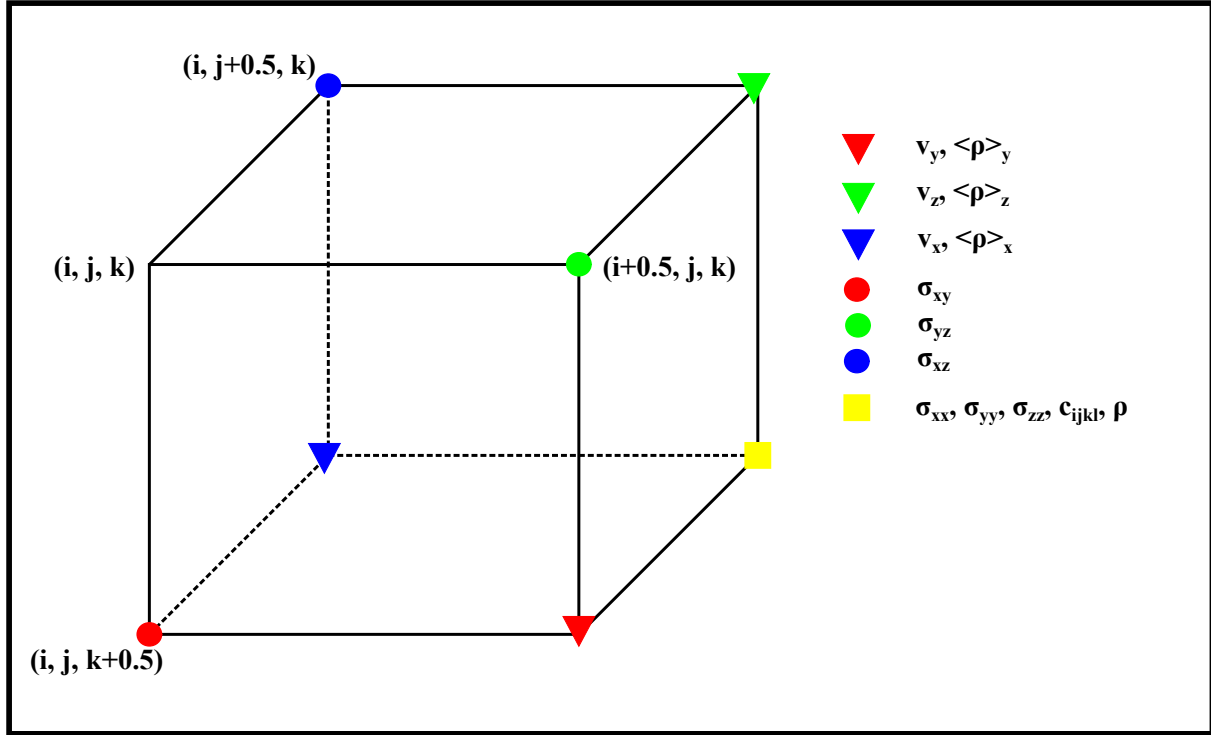


Figure 2.1: Elementary cell for the staggered grid scheme in Cartesian coordinates.

## 2.2 Solution of the elastic wave equation by finite differences

### 2.2.1 Discretization of the equations of motion

For the numerical solution the elastic equations of motion (2.4)-(2.6) are discretized in time and space on an equidistant grid. The particle velocities  $\mathbf{v}_i$ , the stresses  $\sigma_{ij}$  and components of the elastic tensor  $c_{ijkl}$  are defined at discrete Cartesian coordinates  $x = i \, dh$ ,  $y = j \, dh$ ,  $z = k \, dh$  and discrete times  $t = n \, dt$ .  $dh$  denotes the spatial distance between two adjacent grid points and  $dt$  the difference between two successive time steps. Therefore every grid point is located in the interval  $i \in \mathbb{N}[1, NX]$ ,  $j \in \mathbb{N}[1, NY]$ ,  $k \in \mathbb{N}[1, NZ]$  and  $n \in \mathbb{N}[1, NT]$ , where  $NX$ ,  $NY$ ,  $NZ$  and  $NT$  are the maximum number of discrete spatial grid points in each direction and time steps, respectively. Finally the partial and temporal derivatives are replaced by 2nd order finite-difference (FD) operators. Two types of operators can be distinguished, forward and backward operators  $D^+$ ,  $D^-$ . The derivative of a function  $f(x, y, z)$  with respect to the  $x$ -direction can be approximated by the following operators:

$$\begin{aligned} D_x^+ f(x, y, z) &= \frac{f[k, j, i+1] - f[k, j, i]}{dh} && \text{forward operator} \\ D_x^- f(x, y, z) &= \frac{f[k, j, i] - f[k, j, i-1]}{dh} && \text{backward operator} \end{aligned} \quad (2.7)$$

To calculate the spatial derivatives of the wavefield variables at the correct positions and avoid consequent numerical instabilities, the wavefield variables and model parameters are not placed on the same grid points, but staggered by half of the spatial grid point distance [Virieux, 1986, Levander, 1988]. Fig. 2.1 shows the distribution of the material parameters and wavefield variables on the spatial grid. Forward and backward FD operators are chosen according to the positions of the wavefields on the LHS of eq. (2.4) and (2.5), respectively. For the discretization of the momentum

equation (??) the densities have to be averaged arithmetically (Moczo et al. [2004])

$$\begin{aligned}
\langle \rho[k^+, j^+, i] \rangle_x &= \frac{1}{2} \left( \rho[k^+, j^+, i^+] + \rho[k^+, j^+, i^-] \right) \\
\langle \rho[k^+, j, i^+] \rangle_y &= \frac{1}{2} \left( \rho[k^+, j^+, i^+] + \rho[k^+, j^-, i^+] \right) \\
\langle \rho[k, j^+, i^+] \rangle_z &= \frac{1}{2} \left( \rho[k^+, j^+, i^+] + \rho[k^-, j^+, i^+] \right)
\end{aligned} \tag{2.8}$$

Similar to Levander [1988] we introduced the abbreviation  $(i^+, j^+, k^+, n^+) = (i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2}, n + \frac{1}{2})$  and  $(i^-, j^-, k^-, n^-) = (i - \frac{1}{2}, j - \frac{1}{2}, k - \frac{1}{2}, n - \frac{1}{2})$  to denote positions on the half-staggered spatial and temporal grid points. Applying these rules to eq. (2.4) leads to the following pseudo code for the discrete momentum equation

$$\begin{aligned}
v_x^n[k^+, j^+, i] &= v_x^{n-1}[k^+, j^+, i] + \frac{dt}{dh \langle \rho \rangle_x} \left\{ \sigma_{xx}^{n-}[k^+, j^+, i^+] - \sigma_{xx}^{n-}[k^+, j^+, i^-] \right. \\
&\quad \left. + \sigma_{xy}^{n-}[k^+, j+1, i] - \sigma_{xy}^{n-}[k^+, j, i] + \sigma_{xz}^{n-}[k+1, j^+, i] - \sigma_{xz}^{n-}[k, j^+, i] \right\} \\
v_y^n[k^+, j, i^+] &= v_y^{n-1}[k^+, j, i^+] + \frac{dt}{dh \langle \rho \rangle_y} \left\{ \sigma_{xy}^{n-}[k^+, j, i+1] - \sigma_{xy}^{n-}[k^+, j, i] \right. \\
&\quad \left. + \sigma_{yy}^{n-}[k^+, j^+, i^+] - \sigma_{yy}^{n-}[k^+, j^-, i^+] + \sigma_{yz}^{n-}[k+1, j, i^+] - \sigma_{yz}^{n-}[k, j, i^+] \right\} \\
v_z^n[k, j^+, i^+] &= v_z^{n-1}[k, j^+, i^+] + \frac{dt}{dh \langle \rho \rangle_z} \left\{ \sigma_{xz}^{n-}[k, j^+, i+1] - \sigma_{xz}^{n-}[k, j^+, i] \right. \\
&\quad \left. + \sigma_{yz}^{n-}[k, j+1, i^+] - \sigma_{yz}^{n-}[k, j, i^+] + \sigma_{zz}^{n-}[k^+, j^+, i^+] - \sigma_{zz}^{n-}[k^-, j^+, i^+] \right\}.
\end{aligned} \tag{2.9}$$

For the update of the stress-strainrate relationship eq. (2.5) we first discretize the strainrate tensor at the positions of the stress tensor components on the staggered grid

$$\begin{aligned}
\dot{\epsilon}_{xx}[k^+, j^+, i^+] &= \frac{\partial v_x}{\partial x}[k^+, j^+, i^+] \\
\dot{\epsilon}_{yy}[k^+, j^+, i^+] &= \frac{\partial v_y}{\partial y}[k^+, j^+, i^+] \\
\dot{\epsilon}_{zz}[k^+, j^+, i^+] &= \frac{\partial v_z}{\partial z}[k^+, j^+, i^+] \\
\dot{\epsilon}_{yz}[k, j, i^+] &= \frac{1}{2} \left( \frac{\partial v_y}{\partial z}[k, j, i^+] + \frac{\partial v_z}{\partial y}[k, j, i^+] \right) \\
\dot{\epsilon}_{xz}[k, j^+, i] &= \frac{1}{2} \left( \frac{\partial v_z}{\partial x}[k, j^+, i] + \frac{\partial v_x}{\partial z}[k, j^+, i] \right) \\
\dot{\epsilon}_{xy}[k^+, j, i] &= \frac{1}{2} \left( \frac{\partial v_x}{\partial y}[k^+, j, i] + \frac{\partial v_y}{\partial x}[k^+, j, i] \right)
\end{aligned} \tag{2.10}$$

with the FD operators

$$\begin{aligned}
\frac{\partial v_x}{\partial x}[k^+, j^+, i^+] &\approx (v_x[k^+, j^+, i+1] - v_x[k^+, j^+, i])/dh \\
\frac{\partial v_y}{\partial y}[k^+, j^+, i^+] &\approx (v_y[k^+, j+1, i^+] - v_y[k^+, j, i^+])/dh \\
\frac{\partial v_z}{\partial z}[k^+, j^+, i^+] &\approx (v_z[k+1, j^+, i^+] - v_z[k, j^+, i^+])/dh \\
\frac{\partial v_x}{\partial y}[k^+, j, i] &\approx (v_x[k^+, j^+, i] - v_x[k^+, j^-, i])/dh \\
\frac{\partial v_y}{\partial x}[k^+, j, i] &\approx (v_y[k^+, j, i^+] - v_y[k^+, j, i^-])/dh \\
\frac{\partial v_z}{\partial x}[k, j^+, i] &\approx (v_z[k, j^+, i^+] - v_z[k, j^+, i^-])/dh \\
\frac{\partial v_x}{\partial z}[k, j^+, i] &\approx (v_x[k^+, j^+, i] - v_x[k^-, j^+, i])/dh \\
\frac{\partial v_y}{\partial z}[k, j, i^+] &\approx (v_y[k^+, j, i^+] - v_y[k^-, j, i^+])/dh \\
\frac{\partial v_z}{\partial y}[k, j, i^+] &\approx (v_z[k, j^+, i^+] - v_z[k, j^-, i^+])/dh
\end{aligned} \tag{2.11}$$

Substitution of these expressions in the stress-strain-rate relationship eq. (2.5) for the general anisotropic case leads to

$$\begin{aligned}
\sigma_{xx}^{n+}[k^+, j^+, i^+] &= \sigma_{xx}^{n-}[k^+, j^+, i^+] + dt \{ \dot{\epsilon}_{xx}^n c_{11} + \dot{\epsilon}_{yy}^n c_{12} + \dot{\epsilon}_{zz}^n c_{13} \\
&\quad + 2(\dot{\epsilon}_{xy}^{a,n}[k^+, j^+, i^+] c_{16} + \dot{\epsilon}_{xz}^{a,n}[k^+, j^+, i^+] c_{15} + \dot{\epsilon}_{yz}^{a,n}[k^+, j^+, i^+] c_{14}) \} \\
\sigma_{yy}^{n+}[k^+, j^+, i^+] &= \sigma_{yy}^{n-}[k^+, j^+, i^+] + dt \{ \dot{\epsilon}_{xx}^n c_{12} + \dot{\epsilon}_{yy}^n c_{22} + \dot{\epsilon}_{zz}^n c_{23} \\
&\quad + 2(\dot{\epsilon}_{xy}^{a,n}[k^+, j^+, i^+] c_{62} + \dot{\epsilon}_{xz}^{a,n}[k^+, j^+, i^+] c_{52} + \dot{\epsilon}_{yz}^{a,n}[k^+, j^+, i^+] c_{24}) \} \\
\sigma_{zz}^{n+}[k^+, j^+, i^+] &= \sigma_{zz}^{n-}[k^+, j^+, i^+] + dt \{ \dot{\epsilon}_{xx}^n c_{13} + \dot{\epsilon}_{yy}^n c_{23} + \dot{\epsilon}_{zz}^n c_{33} \\
&\quad + 2(\dot{\epsilon}_{xy}^{a,n}[k^+, j^+, i^+] c_{63} + \dot{\epsilon}_{xz}^{a,n}[k^+, j^+, i^+] c_{53} + \dot{\epsilon}_{yz}^{a,n}[k^+, j^+, i^+] c_{43}) \}
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\sigma_{xy}^{n+}[k^+, j, i] &= \sigma_{xy}^{n-}[k^+, j, i] + dt \{ \dot{\epsilon}_{xx}^{a,n}[k^+, j, i] c_{16}^h[k^+, j, i] + \dot{\epsilon}_{yy}^{a,n}[k^+, j, i] c_{62}^h[k^+, j, i] + \dot{\epsilon}_{zz}^{a,n}[k^+, j, i] c_{63}^h[k^+, j, i] \\
&\quad + 2(\dot{\epsilon}_{xy}^n[k^+, j, i] c_{66}^h[k^+, j, i] + \dot{\epsilon}_{xz}^{a,n}[k^+, j, i] c_{65}^h[k^+, j, i] + \dot{\epsilon}_{yz}^{a,n}[k^+, j, i] c_{64}^h[k^+, j, i]) \} \\
\sigma_{xz}^{n+}[k, j^+, i] &= \sigma_{xz}^{n-}[k, j^+, i] + dt \{ \dot{\epsilon}_{xx}^{a,n}[k, j^+, i] c_{15}^h[k, j^+, i] + \dot{\epsilon}_{yy}^{a,n}[k, j^+, i] c_{52}^h[k, j^+, i] + \dot{\epsilon}_{zz}^{a,n}[k, j^+, i] c_{53}^h[k, j^+, i] \\
&\quad + 2(\dot{\epsilon}_{xy}^{a,n}[k, j^+, i] c_{65}^h[k, j^+, i] + \dot{\epsilon}_{xz}^n[k, j^+, i] c_{55}^h[k, j^+, i] + \dot{\epsilon}_{yz}^{a,n}[k, j^+, i] c_{54}^h[k, j^+, i]) \} \\
\sigma_{yz}^{n+}[k, j, i^+] &= \sigma_{yz}^{n-}[k, j, i^+] + dt \{ \dot{\epsilon}_{xx}^{a,n}[k, j, i^+] c_{14}^h[k, j, i^+] + \dot{\epsilon}_{yy}^{a,n}[k, j, i^+] c_{24}^h[k, j, i^+] + \dot{\epsilon}_{zz}^{a,n}[k, j, i^+] c_{43}^h[k, j, i^+] \\
&\quad + 2(\dot{\epsilon}_{xy}^{a,n}[k, j, i^+] c_{64}^h[k, j, i^+] + \dot{\epsilon}_{xz}^{a,n}[k, j, i^+] c_{54}^h[k, j, i^+] + \dot{\epsilon}_{yz}^n[k, j, i^+] c_{44}^h[k, j, i^+]) \}
\end{aligned} \tag{2.13}$$

To simplify the expression of the elastic tensor we introduced the Voigt notation [Voigt, 1910], where pairs of tensor indices are related to integer numbers from 1 to 6:  $(1, 1) \rightarrow 1$ ,  $(2, 2) \rightarrow 2$ ,  $(3, 3) \rightarrow 3$ ,  $(2, 3) \rightarrow 4$ ,  $(1, 3) \rightarrow 5$ ,  $(1, 2) \rightarrow 6$ . The correct update of the stress tensor components on the staggered grid requires the arithmetic averages of certain strain-rate tensor components  $\dot{\epsilon}_{ij}^a$  and the harmonic averages of elastic tensor components  $c_{ij}^h$ . The details of the averaging are described in appendix A. A detailed dispersion and stability analysis for the isotropic case can be found in Crase [1990], Igel et al. [1995], Saenger et al. [2000], Saenger and Bohlen [2004]. For the general anisotropic case Igel et al. [1995] suggests to replace the maximum P-wave velocity of the isotropic medium in the stability criterion by the maximum phase velocity in the anisotropic medium. In order to model wave propagation in an elastic full- or half-space convolutional PML (C-PML) absorbing boundary conditions are implemented according to the approach by [Komatitsch and Martin, 2007]. To reduce computation time, the resulting code is parallelized by domain decomposition using MPI.

### 2.2.2 Accuracy of FD operators

The derivation of the FD operators in the last section was a simple replacement of the partial derivatives by finite differences. In the following more systematic approach, the first derivative of a variable  $f$  at a grid point  $i$  is estimated by a Taylor series expansion (Jastram [1992]):

$$(2k-1) \frac{\partial f}{\partial x} \Big|_i = \frac{1}{dh} (f_{i+(k-1/2)} - f_{i-(k-1/2)}) + \frac{1}{dh} \sum_{l=2}^N \frac{((k-\frac{1}{2})dh)^{2l-1}}{(2l-1)!} \frac{\partial^{(2l-1)} f}{\partial x^{(2l-1)}} \Big|_i + \mathcal{O}(dh)^{2N}$$

For an operator with length  $2N$ ,  $N$  equations are added with a weight  $\beta_k$ :

$$\begin{aligned} \left[ \sum_{k=1}^N \beta_k (2k-1) \right] \frac{\partial f}{\partial x} \Big|_i &= \frac{1}{dh} \sum_{k=1}^N \beta_k (f_{i+(k-1/2)} - f_{i-(k-1/2)}) \\ &+ \frac{1}{dh} \sum_{k=1}^N \sum_{l=2}^N \beta_k \frac{((k-\frac{1}{2})dh)^{2l-1}}{(2l-1)!} \frac{\partial^{(2l-1)} f}{\partial x^{(2l-1)}} \Big|_i + \mathcal{O}(dh)^{2N} \end{aligned} \quad (2.14)$$

The case  $N=1$  leads to the FD operator derived in the last section, which has a length of  $2N=2$ . The Taylor series is truncated after the first term ( $\mathcal{O}(dh)^2$ ). Therefore this operator is called **2nd order FD operator** which refers to the truncation error of the Taylor series and not to the order of the approximated derivative. To understand equation (2.14) better, we estimate a **4th order FD operator**. This operator has the length  $2N = 4$  or  $N=2$ . The sums in Eq. (2.14) lead to:

$$\begin{aligned} (\beta_1 + 3\beta_2) \frac{\partial f}{\partial x} \Big|_i &= \frac{1}{dh} (\beta_1 (f_{i+1/2} - f_{i-1/2}) + \beta_2 (f_{i+3/2} - f_{i-3/2})) \\ &+ \frac{dh^3}{dh} \left[ \beta_1 \frac{1}{8 \cdot 3!} + \beta_2 \frac{27}{8 \cdot 3!} \right] \frac{\partial^3 f}{\partial x^3} \Big|_i \end{aligned} \quad (2.15)$$

The weights  $\beta_k$  can be calculated by the following approach: The factor in front of the partial derivative on the LHS of Eq. (2.15) should equal 1, therefore

$$(\beta_1 + 3\beta_2) = 1.$$

The coefficients in front of  $\frac{\partial^3 f}{\partial x^3} \Big|_i$  on the RHS of Eq. (2.15) should vanish:

$$(\beta_1 + 27\beta_2) = 0.$$

The weights  $\beta_k$  can be estimated by solving the matrix equation:

$$\begin{pmatrix} 1 & 3 \\ 1 & 27 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The resulting coefficients are  $\beta_1 = 9/8$  and  $\beta_2 = -1/24$ . Therefore the 4th order backward- and forward operators are:

$$\begin{aligned} \frac{\partial f}{\partial x} \Big|_{i+1/2} &= \frac{1}{dh} [\beta_1 (f_{i+1} - f_i) + \beta_2 (f_{i+2} - f_{i-1})] && \text{forward operator} \\ \frac{\partial f}{\partial x} \Big|_{i-1/2} &= \frac{1}{dh} [\beta_1 (f_i - f_{i-1}) + \beta_2 (f_{i+1} - f_{i-2})] && \text{backward operator} \end{aligned} \quad (2.16)$$

The coefficients  $\beta_i$  in the FD operator are called **Taylor coefficients**. The accuracy of higher order FD operators can be improved by seeking for FD coefficients  $\beta_k$  that approximate the first derivative in a certain frequency range (Holberg [1987]). These numerically optimized coefficients are called **Holberg coefficients**.

### 2.2.3 Initial and Boundary Conditions

To find a unique solution of the problem, initial and boundary conditions have to be defined. The initial conditions for the elastic forward problem are:

$$\begin{aligned} u_i(\mathbf{x}, t) &= 0 \\ \frac{\partial u_i(\mathbf{x}, t)}{\partial t} &= 0 \end{aligned} \quad (2.17)$$

for all  $\mathbf{x} \in V$  at  $t = 0$ .

For the geophysical application two types of boundary conditions are very important:

1. **Horizontal Free Surface:** The interface between the elastic medium and air at the surface is very important when trying to model surface waves or multiple reflections in a marine environment. Since all stresses in the normal direction at this interface vanish

$$\sigma_{xy} = \sigma_{yy} = 0.0 \quad (2.18)$$

this boundary condition is called (stress) **free surface**. Two types of implementations are common. In the implicit definition of the free surface, a small layer with the acoustic parameters of air ( $V_p = 300$  m/s,  $V_s = 0.0$  m/s,  $\rho = 1.25$  kg/m<sup>3</sup>) is placed on top of the model. One advantage of the implicit definition of the free surface is the easy implementation of topography on the FD grid, however to get accurate results for surface waves or multiples, this approach requires a fine spatial sampling of the FD grid near the free surface. An explicit free surface can be implemented by using the mirroring technique by Levander, which leads to stable and accurate solutions for plain interfaces (Levander [1988], Robertsson et al. [1995]). If the planar free surface is located at grid point  $j = h$ , the stress at this point is set to zero and the stresses below the free surface are mirrored with an inverse sign:

$$\begin{aligned} \sigma_{yy}(h, i) &= 0 \\ \sigma_{yy}(h-1, i) &= -\sigma_{yy}(h+1, i) \\ \sigma_{xy}(h-\frac{1}{2}, i+\frac{1}{2}) &= -\sigma_{xy}(h+\frac{1}{2}, i+\frac{1}{2}) \\ \sigma_{xy}(h-\frac{3}{2}, i+\frac{1}{2}) &= -\sigma_{xy}(h+\frac{3}{2}, i+\frac{1}{2}) \end{aligned} \quad (2.19)$$

When updating the stress component  $\sigma_{xx} = (\lambda + 2\mu)u_{xx} + \lambda u_{yy}$  at the free surface, only horizontal particle displacements should be used because vertical derivatives over the free surface lead to instabilities (Levander [1988]). The vertical derivative of the y-displacement  $u_{yy}$  can be replaced by using the boundary condition at the free surface:

$$\begin{aligned} \sigma_{yy} &= (\lambda + 2\mu)u_{yy} + \lambda u_{xx} = 0 \\ u_{yy} &= -\frac{\lambda}{(\lambda + 2\mu)}u_{xx} \end{aligned} \quad (2.20)$$

Therefore the stress  $\sigma_{xx}$  can be written as

$$\sigma_{xx} = \frac{4(\lambda\mu + \mu^2)}{\lambda + 2\mu}u_{xx} \quad (2.21)$$

2. **Absorbing Boundary Conditions:** Due to limited computational resources, the FD grid has to be as small as possible. To model problems with an infinite extension in different directions, e.g. a full or half-space problem, an artificial absorbing boundary condition has to be applied. A very effective way to damp the waves near the boundaries are **Perfectly Matched Layers (PMLs)**. This can be achieved by a coordinate stretch of the wave equations in the frequency domain (Komatitsch and Martin [2007]). The coordinate stretch creates exponentially decaying plane wave solutions in the absorbing boundary frame. The PML's are only reflectionless if the exact wave equation is solved. As soon as the problem is discretized (for example using finite differences) you are solving an approximate wave equation and the analytical perfection of the PML is no longer valid. To overcome this shortcoming the wavefield is damped by the damping function

$$c = -V_{pml} * \frac{\log(\alpha)}{L} \quad (2.22)$$

where  $V_{\text{pml}}$  denotes the typical P-wave velocity of the medium in the absorbing boundary frame,  $\alpha = 1 \times 10^{-4}$  and  $L$  is the thickness of the absorbing boundary layer. A comparison between the exponential damping and the PML boundary is shown in Fig.2.2. The PMLs are damping the seismic waves by a factor 5-10 more effective than the absorbing boundary frame.

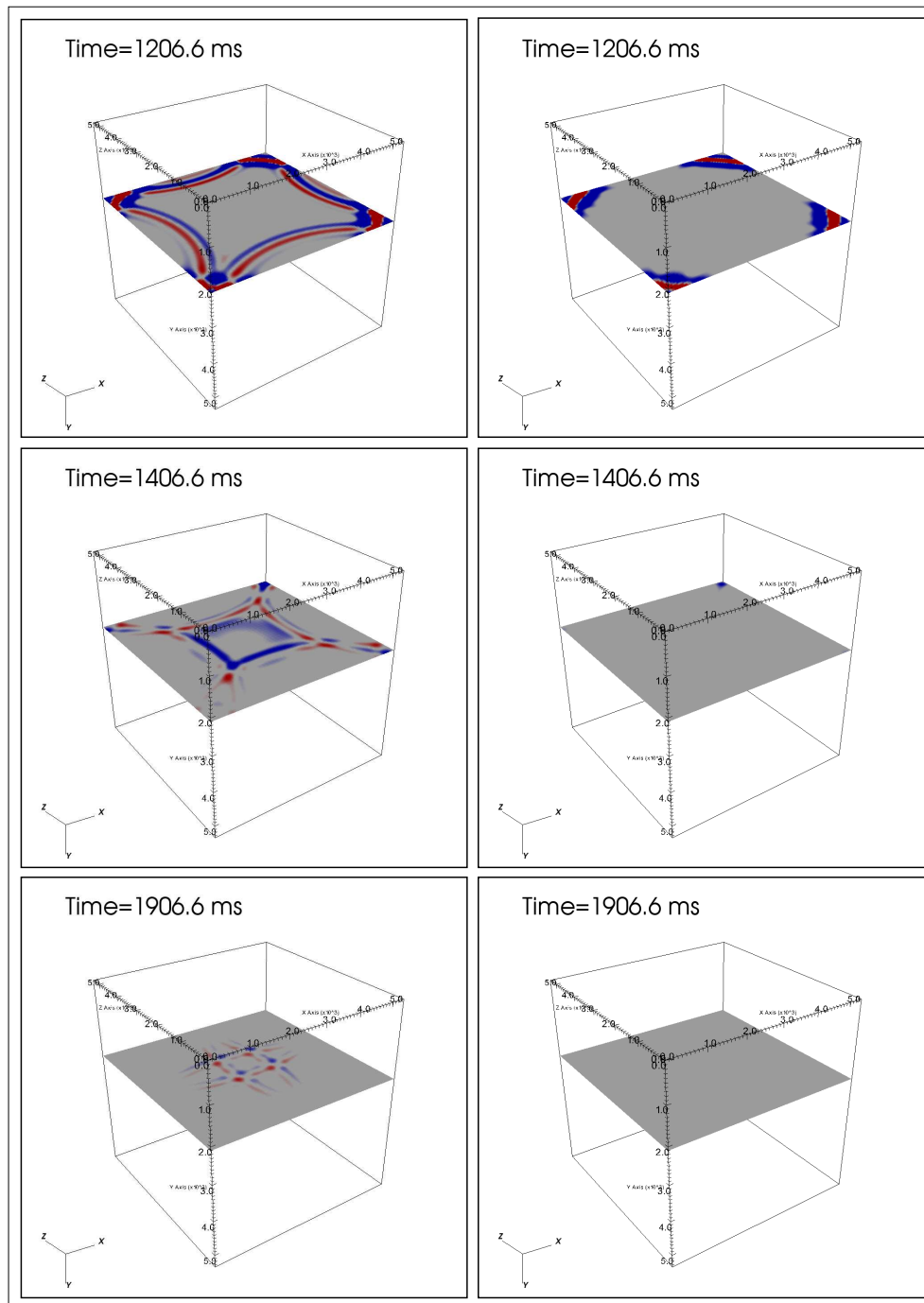


Figure 2.2: Comparison between exponential damping (left column) and PML (right column) absorbing boundary conditions for a homogeneous full space model.

## 2.3 Numerical Artefacts and Instabilities

To avoid numerical artefacts and instabilities during a FD modelling run, spatial and temporal sampling conditions for the wavefield have to be satisfied. These will be discussed in the following two sections.

### 2.3.1 Grid Dispersion

The first question when building a FD model is: What is the maximum spatial grid point distance  $dh$ , for a correct sampling of the wavefield? To answer this question we take a look at this simple example: The particle displacement in  $x$ -direction is defined by a sine function:

$$u_x = \sin\left(2\pi \frac{x}{\lambda}\right), \quad (2.23)$$

where  $\lambda$  denotes the wavelength. When calculating the derivation of this function analytically at  $x = 0$  and setting  $\lambda = 1$  m we get:

$$\left. \frac{du_x}{dx} \right|_{x=0} = \frac{2\pi}{\lambda} \cos\left(2\pi \frac{x}{\lambda}\right) \Big|_{x=0} = 2\pi. \quad (2.24)$$

In the next step the derivation is approximated numerically by a staggered 2nd order finite-difference operator:

$$\left. \frac{du_x}{dx} \right|_{x=0} \approx \frac{u_x(x + \frac{1}{2}\Delta x) - u_x(x - \frac{1}{2}\Delta x)}{\Delta x} \Big|_{x=0} = \frac{\sin\left(\frac{2\pi(x + \frac{1}{2}\Delta x)}{\lambda}\right) - \sin\left(\frac{2\pi(x - \frac{1}{2}\Delta x)}{\lambda}\right)}{\Delta x}. \quad (2.25)$$

Using the Nyquist-Shannon sampling theorem it should be sufficient to sample the wavefield with  $\Delta x = \lambda/2$ . In table 2.1 the numerical solutions of eq. (2.25) and the analytical solution (2.24) are compared for different sample intervals  $\Delta x = \lambda/n$ , where  $n$  is the number of gridpoints per wavelength. For the case  $n=2$ , which corresponds to the Nyquist-Shannon theorem, the numerical solution is  $\left. \frac{du_x}{dx} \right|_{x=0} = 4.0$ , which is not equal with the analytical solution  $2\pi$ . A refinement of the spatial sampling of the wavefield results in an improvement of the finite difference solution. For  $n = 16$  the numerical solution is accurate to the second decimal place. The effect of a sparsely sampled pressure field is illustrated in figure 2.3 for a homogeneous block model with stress free surfaces. The dimensions of the FD grid are fixed and the central frequency of the source signal is increased systematically. When using a spatial sampling of 16 grid points per minimum wavelength (figure 2.3, top) the wavefronts are sharply defined. For  $n = 4$  grid points a slight numerical dispersion of the wave occurs (figure 2.3, center). This effect is obvious when using the Nyquist criterion ( $n = 2$ ) (figure 2.3, bottom). Since the numerical calculated wavefield seem to be dispersive this numerical artefact is called **grid dispersion**. To avoid the occurrence of grid dispersion the following criteria for the spatial grid spacing  $dh$  has to be satisfied:

$$dh \leq \frac{\lambda_{\min}}{n} = \frac{V_{\min}}{n f_{\max}}. \quad (2.26)$$

Here  $\lambda_{\min}$  denotes the minimum wavelength,  $V_{\min}$  the minimum velocity in the model and  $f_{\max}$  is the maximum frequency of the source signal. Depending on the accuracy of the used FD operator the parameter  $n$  is different. In table 2.2  $n$  is listed for different FD operator lengths and types (Taylor and Holberg operators). The Holberg coefficients are calculated for a minimum dispersion error of 0.1% at  $3f_{\max}$ . For short operators  $n$  should be chosen relatively large, so the spatial grid spacing is small, while for longer FD operators  $n$  is smaller and the grid spacing can be larger.

n	$\Delta x$ [m]	$\frac{dv_x}{dx} _{x=0}$ [ ]
analytical	-	$2\pi \approx 6.283$
2	$\lambda/2$	4.0
4	$\lambda/4$	5.657
8	$\lambda/8$	6.123
16	$\lambda/16$	6.2429
32	$\lambda/32$	6.2731

Table 2.1: Comparison of the analytical solution Eq. (2.24) with the numerical solution Eq. (2.25) for different grid spacings  $\Delta x = \lambda/n$ .

FDORDER	n (Taylor)	n (Holberg)
2nd	12	12
4th	8	8.32
6th	6	4.77
8th	5	3.69
10th	5	3.19
12th	4	2.91

Table 2.2: The number of grid points per minimum wavelength  $n$  for different orders (2nd-12th) and types (Taylor and Holberg) of FD operators. For the Holberg coefficients  $n$  is calculated for a minimum dispersion error of 0.1% at  $3f_{\max}$ .

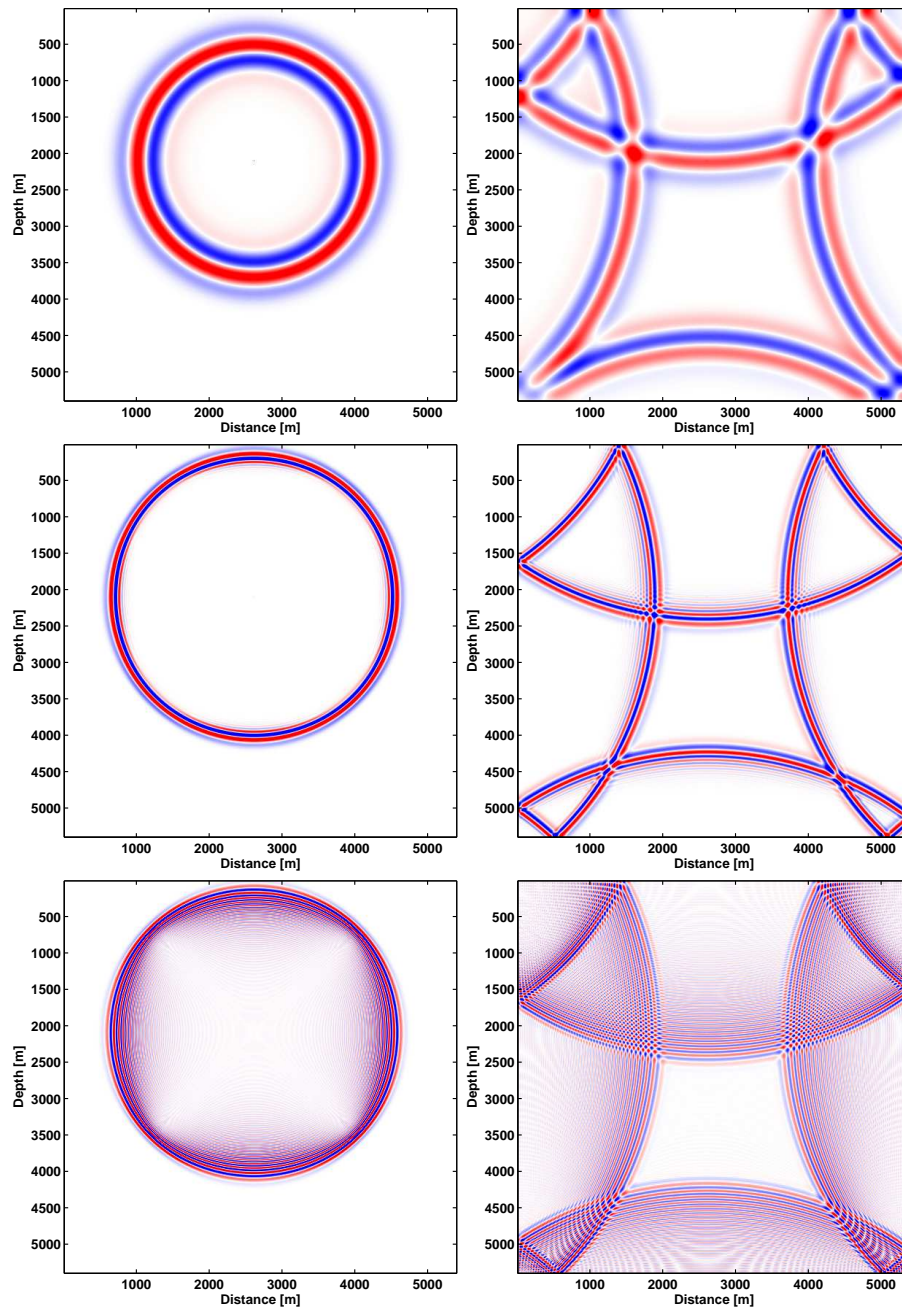


Figure 2.3: The influence of grid dispersion in FD modelling: Spatial sampling of the wavefield using  $n=16$  (top),  $n=4$  (center) and  $n=2$  gridpoints (bottom) per minimum wavelength  $\lambda_{\min}$ .

### 2.3.2 The Courant Instability

Beside the spatial, the temporal discretization has to satisfy a sampling criterion to ensure the stability of the FD code. If a wave is propagating on a discrete grid, then the timestep  $dt$  has to be less than the time for the wave to travel between two adjacent grid points with grid spacing  $dh$ . For an elastic 2D grid this means mathematically:

$$dt \leq \frac{dh}{h\sqrt{2}V_{\max}}, \quad (2.27)$$

where  $V_{\max}$  is the maximum velocity in the model. The factor  $h$  depends on the order of the FD operator and can easily be calculated by summing over the weighting coefficients  $\beta_i$

$$h = \sum_i \beta_i. \quad (2.28)$$

In table 2.3  $h$  is listed for different FD operator lengths and types (Taylor and Holberg operators). Criterion (2.27) is called **Courant-Friedrichs-Lewy criterion** (Courant et al. [1928], Courant et al. [March 1967]). figure 2.4 shows the evolution of the pressure field when the Courant criterion is violated. After a few time steps the amplitudes are growing to infinity and the calculation becomes unstable.

FDORDER	$h$ (Taylor)	$h$ (Holberg)
2nd	1.0	1.0
4th	7/6	1.184614
6th	149/120	1.283482
8th	2161/1680	1.345927
10th	53089/40320	1.387660
12th	1187803/887040	1.417065

Table 2.3: The factor  $h$  in the Courant criterion for different orders (2nd-12th) and types (Taylor and Holberg) of FD operators.

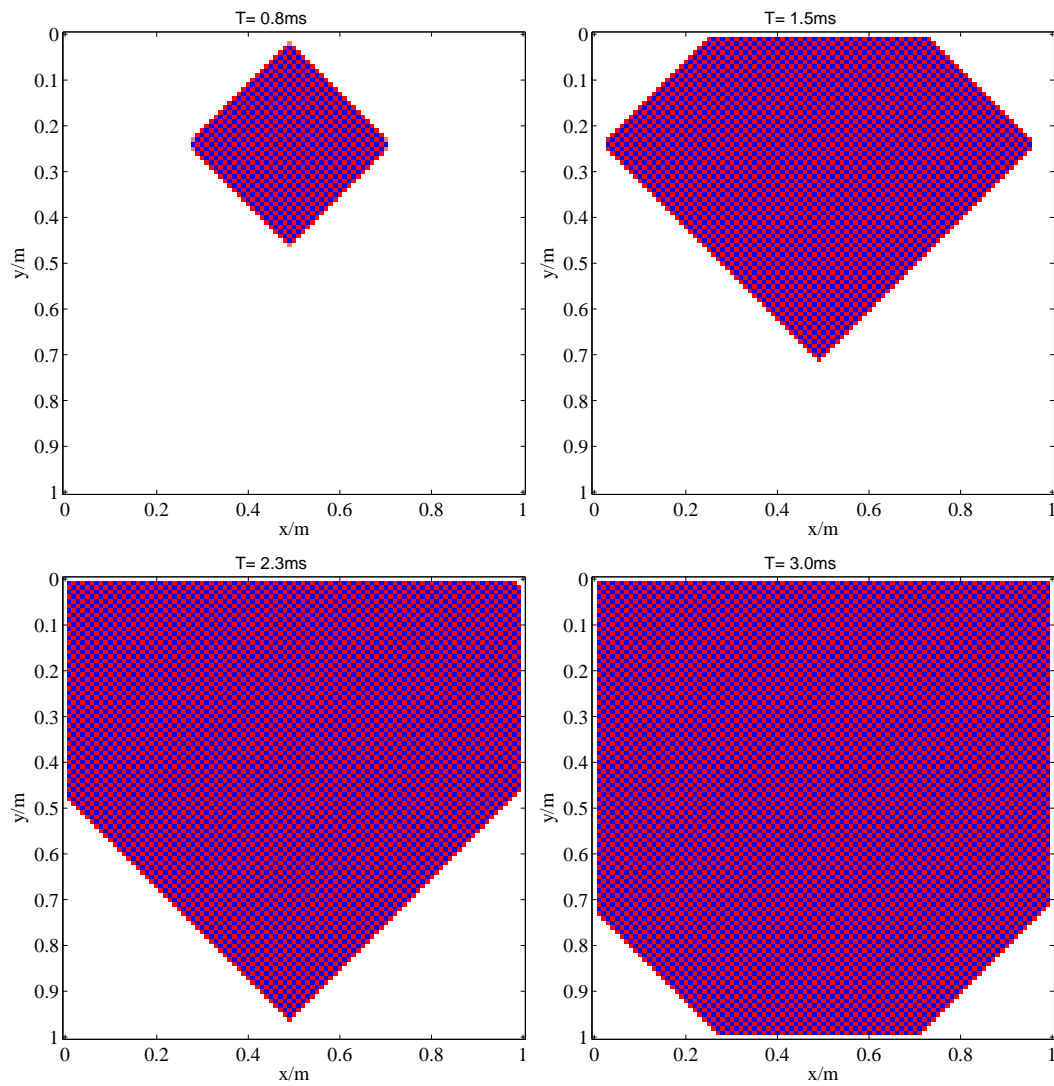


Figure 2.4: Temporal evolution of the Courant instability. In the colored areas the wave amplitudes are extremely large.

## Chapter 3

# The adjoint problem

The aim of full waveform tomography is to find an "optimum" model which can explain the data very well. It should not only explain the first arrivals of specific phases of the seismic wavefield like refractions or reflections, but also the amplitudes which contain information on the distribution of the elastic material parameters in the underground. To achieve this goal three problems have to be solved:

1. What is an "optimum" model ?
2. How can this model be found ?
3. Is this model unique or are other models existing, which could explain the data equally well ?

### 3.1 What is an "optimum" model ?

In reflection seismics the  $i^{\text{th}}$  component of the elastic displacement field  $u_i(\mathbf{x}_s, \mathbf{x}_r, t)$  excited by sources located at  $\mathbf{x}_s$  will be recorded by receivers at  $\mathbf{x}_r$  at time  $t$ . For a given distribution of the material parameters the forward problem Eq. 2.3 can be solved by finite differences (section 2.2). The result is a model data set  $\mathbf{u}^{\text{mod}}$ . This modelled data can be compared with the field data  $\mathbf{u}^{\text{obs}}$ . If the misfit or data residuals  $\delta\mathbf{u} = \mathbf{u}^{\text{mod}} - \mathbf{u}^{\text{obs}}$  (figure 3.1) between the modelled and the field data is small the model can explain the data very well. If the residuals are large the model cannot explain the data. The misfit can be measured by a vector norm  $|\mathbf{L}|_p$  which is defined for  $p = 1, 2, \dots$  as

$$|\mathbf{L}|_p = \left( \sum_i |\delta u_i|^p \right)^{1/p} \quad (3.1)$$

The special case  $|\mathbf{L}|_\infty$  is defined as

$$|\mathbf{L}|_\infty = \max_i |\delta u_i| \quad (3.2)$$

The L2-norm

$$E = |\mathbf{L}|_2 = \frac{1}{2} \delta \mathbf{u}^T \delta \mathbf{u} \quad (3.3)$$

has a special physical meaning. It represents the residual elastic energy contained in the data residuals  $\delta\mathbf{u}$ . An optimum model can be found in a minimum of the residual energy. Therefore the optimum model is the solution of a nonlinear optimization problem.

### 3.2 How to find an optimum model

Figure 3.2 shows a schematic sketch of the residual energy at one point in space as a function of two model parameters  $\lambda$  and  $\mu$ . The colors represent different values of the residual energy. Red areas represent models with high residual energy which do not fit the data, while the blue parts are good fitting models with low residual energies. The aim is to find the minimum of the residual energy marked by the red cross. Starting at a point  $\mathbf{m}_1 = (\lambda_1(\mathbf{x}), \mu_1(\mathbf{x}), \rho_1(\mathbf{x}), \dots)$  in the parameter space we want to find the minimum by updating the material parameters in an iterative way

$$\mathbf{m}_2 = \mathbf{m}_1 + \mu_1 \delta \mathbf{m}_1, \quad (3.4)$$

along the search direction  $\delta \mathbf{m}_1$  with the step length  $\mu_1$ . To find the optimum search direction  $\delta \mathbf{m}_1$  we expand the residual energy  $E(\mathbf{m}_1 + \delta \mathbf{m}_1)$  near the starting point in a Taylor series:

$$E(\mathbf{m}_1 + \delta \mathbf{m}_1) \approx E(\mathbf{m}_1) + \delta \mathbf{m}_1 \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1 + \frac{1}{2} \delta \mathbf{m}_1 \left( \frac{\partial^2 E}{\partial \mathbf{m}^2} \right)_1 \delta \mathbf{m}_1^T \quad (3.5)$$

and set the derivative of Eq. 3.5 with respect to  $\delta \mathbf{m}_1$  zero

$$\frac{\partial E(\mathbf{m}_1 + \delta \mathbf{m}_1)}{\partial \delta \mathbf{m}_1} = \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1 + \delta \mathbf{m}_1 \left( \frac{\partial^2 E}{\partial \mathbf{m}^2} \right)_1 = 0 \quad (3.6)$$

Which finally leads to

$$\delta \mathbf{m}_1 = - \left( \frac{\partial^2 E}{\partial \mathbf{m}^2} \right)_1^{-1} \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1 = -\mathbf{H}_1^{-1} \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1 \quad (3.7)$$

where  $(\partial E / \partial \mathbf{m})_1$  denotes the steepest-descent direction of the objective function and  $\mathbf{H}_1^{-1}$  the inverse Hessian matrix. The inverse Hessian matrix for the elastic problem is often singular and can only be calculated with high computational costs. Therefore the inverse Hessian matrix is approximated by a preconditioning operator  $\mathbf{P}$ . There is

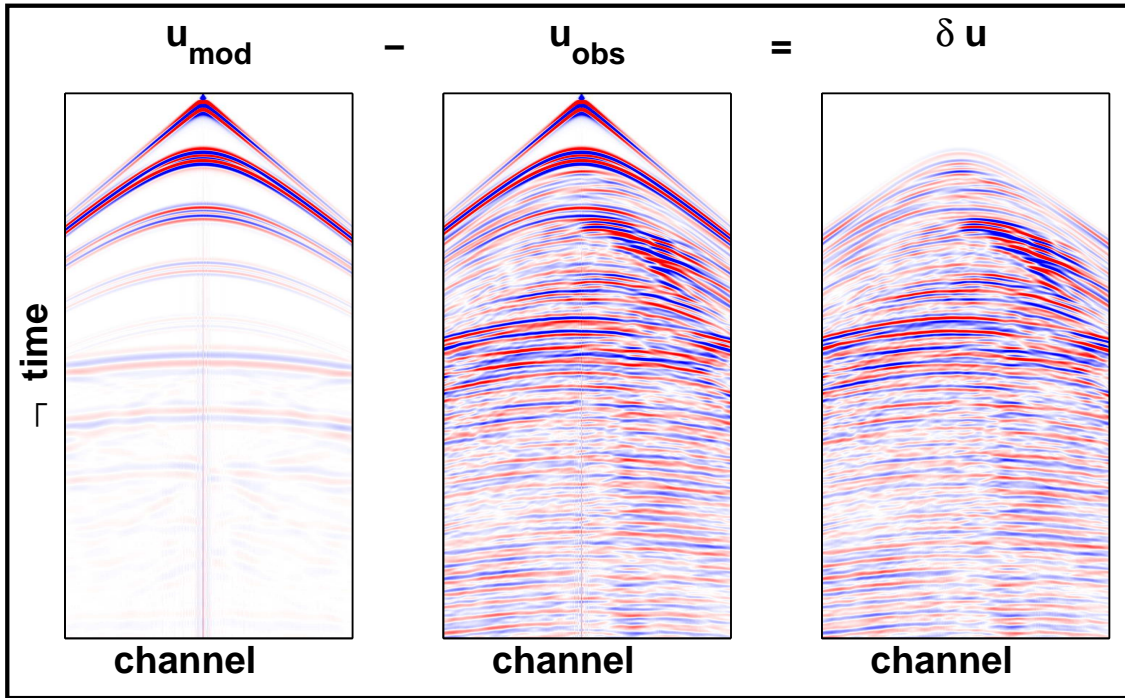


Figure 3.1: Definition of data residuals  $\delta \mathbf{u}$ .

no general rule for an optimum preconditioning operator, but two very simple operators are described in more detail in chapter ?? for a cross-well acquisition geometry and in chapter ?? for a reflection geometry.

$$\delta \mathbf{m}_1 \approx -\mathbf{P}_1 \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1. \quad (3.8)$$

By replacing  $\delta \mathbf{m}_1$  in Eq. 3.4 with Eq. 3.8 we get

$$\mathbf{m}_2 = \mathbf{m}_1 - \mu_1 \mathbf{P}_1 \left( \frac{\partial E}{\partial \mathbf{m}} \right)_1, \quad (3.9)$$

The optimum model parameters can be found along the negative gradient direction of the residual energy. The starting point  $\mathbf{m}_1$  is not a particular point, so the update function can be applied to every point in the parameter space  $\mathbf{m}_n$

$$\mathbf{m}_{n+1} = \mathbf{m}_n - \mu_n \mathbf{P}_n \left( \frac{\partial E}{\partial \mathbf{m}} \right)_n. \quad (3.10)$$

### 3.3 Calculation of the gradient direction $\frac{\partial E}{\partial \mathbf{m}}$

To estimate the gradient direction  $\partial E / \partial \mathbf{m}$  the residual energy is rewritten as:

$$E = \frac{1}{2} \delta \mathbf{u}^T \delta \mathbf{u} = \frac{1}{2} \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \delta \mathbf{u}^2(\mathbf{x}_r, \mathbf{x}_s, t) \quad (3.11)$$

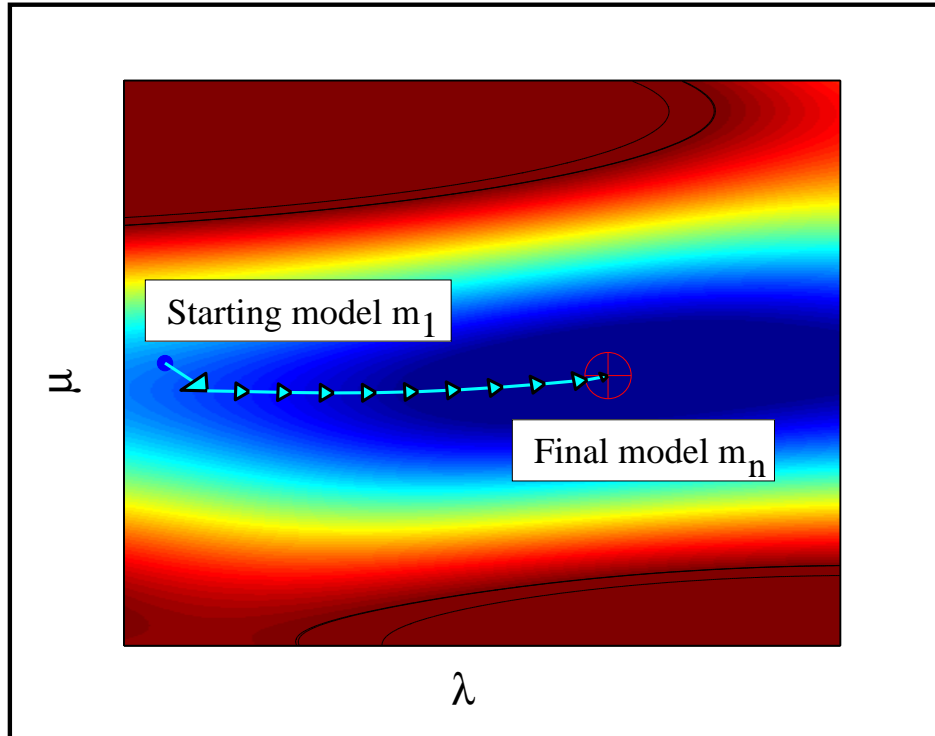


Figure 3.2: Schematic sketch of the residual energy at one point in space as a function of two model parameters  $m_1$  and  $m_2$ . The blue dot denotes the starting point in the parameter space, while the red cross marks a minimum of the objective function.

After derivation with respect to a model parameter  $\mathbf{m}$  we get

$$\begin{aligned}
 \frac{\partial E}{\partial \mathbf{m}} &= \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \frac{\partial \delta \mathbf{u}}{\partial \mathbf{m}} \delta \mathbf{u} \\
 &= \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \frac{\partial (\mathbf{u}^{\text{mod}}(\mathbf{m}) - \mathbf{u}^{\text{obs}})}{\partial \mathbf{m}} \delta \mathbf{u} \\
 &= \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \frac{\partial \mathbf{u}^{\text{mod}}(\mathbf{m})}{\partial \mathbf{m}} \delta \mathbf{u}
 \end{aligned} \tag{3.12}$$

Eq. (3.12) can be related to the mapping of small changes from the data to the model space and vice versa (figure 3.3). A small change in the model space  $\delta \mathbf{m}$ , e.g. one model parameter at one point in space, will result in a small

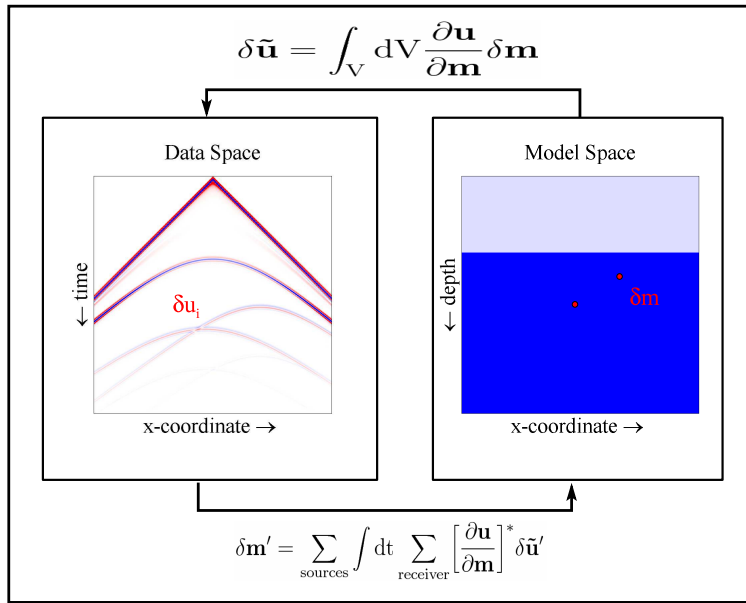


Figure 3.3: Mapping between model and data space and vice versa.

perturbation of the data space  $\delta \tilde{\mathbf{u}}$ , e.g. one wiggle in the seismic section. If the Frechét derivative  $\frac{\partial \mathbf{u}}{\partial \mathbf{m}}$  is known, all the small perturbations in model space can be integrated over the model volume  $V$  to calculate the total change in data space (Tarantola [2005]):

$$\delta \tilde{\mathbf{u}}(\mathbf{x}_s, \mathbf{x}_r, t) = \int_V dV \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \delta \mathbf{m}, \tag{3.13}$$

or by introducing the linear operator  $\hat{L}$

$$\delta \tilde{\mathbf{u}} = \hat{L} \delta \mathbf{m} := \int_V dV \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \delta \mathbf{m}.$$

In a similar way small changes in the data space  $\delta \tilde{\mathbf{u}}'$  can be integrated to calculate the total change in the model space  $\delta \mathbf{m}'$  (Tarantola [2005])

$$\delta \mathbf{m}' = \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \right]^* \delta \tilde{\mathbf{u}}', \tag{3.14}$$

or as operator equation

$$\delta \mathbf{m}' = \hat{L}^* \delta \tilde{\mathbf{u}}'.$$

In this case the Frechét derivative  $\frac{\partial \mathbf{u}}{\partial \mathbf{m}}$  is replaced by its adjoint counterpart  $\frac{\partial \mathbf{u}}{\partial \mathbf{m}}^*$ . Note that  $\delta \tilde{\mathbf{u}} \neq \delta \tilde{\mathbf{u}}'$  and  $\delta \mathbf{m} \neq \delta \mathbf{m}'$ , so there is no unique way to map perturbations from the model to the data space or vice versa. Because the operator  $\hat{L}$  is linear, the kernel of  $\hat{L}$  and its adjoint counterpart  $\hat{L}^*$  are identical (see chapter 5.4.2 in Tarantola [2005])

$$\left[ \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \right]^* = \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \right]$$

Therefore the mapping from the data to the model space Eq. (3.14) is equal to the gradient of the residual energy Eq. (3.12):

$$\begin{aligned} \delta \mathbf{m}' &= \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \left[ \frac{\partial \mathbf{u}_i}{\partial \mathbf{m}} \right]^* \delta \tilde{\mathbf{u}}' \\ &= \sum_{\text{sources}} \int dt \sum_{\text{receiver}} \left[ \frac{\partial \mathbf{u}_i}{\partial \mathbf{m}} \right] \delta \mathbf{u} \\ &= \frac{\partial E}{\partial \mathbf{m}} \end{aligned} \quad (3.15)$$

if the perturbation of the data space  $\delta \tilde{\mathbf{u}}'$  is interpreted as data residuals  $\delta \mathbf{u}$ . So the approach to estimate the gradient direction  $\partial E / \partial \mathbf{m}$  can be split into 3 parts

1. Find a solution to the forward problem

$$\delta \mathbf{u} = \hat{L} \delta \mathbf{m}.$$

2. Identify the Frechét kernels  $\partial \mathbf{u} / \partial \mathbf{m}$

3. Use the property, that a linear operator  $\hat{L}$  and its adjoint  $\hat{L}^*$  have the same kernels and calculate the gradient direction by using:

$$\frac{\partial E}{\partial \mathbf{m}} = \delta \mathbf{m}' = \hat{L}^* \delta \mathbf{u}'.$$

This is a very general approach. Now we apply this approach to the equations of motion for an elastic medium. The elastic forward problem Eqs. (2.3) can be written as

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}_i}{\partial t^2} - \frac{\partial}{\partial x_j} \sigma_{ij} &= f_i, \\ \sigma_{ij} - c_{ijkl} \epsilon_{kl} &= T_{ij}, \\ \epsilon_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \\ &+ \text{initial and boundary conditions,} \end{aligned} \quad (3.16)$$

where  $\rho$  denotes the density,  $\sigma_{ij}$  the stress tensor,  $\epsilon_{ij}$  the strain tensor,  $c_{ijkl}$  the stiffness tensor,  $f_i$ ,  $T_{ij}$  source terms for volume and surface forces, respectively. In the next step every parameter and variable in the elastic wave equation is perturbed by a first order perturbation as shown in Fig. 3.3:

$$\begin{aligned} u_i &\rightarrow u_i + \delta u_i, \\ \rho &\rightarrow \rho + \delta \rho, \\ \sigma_{ij} &\rightarrow \sigma_{ij} + \delta \sigma_{ij}, \\ c_{ijkl} &\rightarrow c_{ijkl} + \delta c_{ijkl}, \\ \epsilon_{ij} &\rightarrow \epsilon_{ij} + \delta \epsilon_{ij}, \end{aligned} \quad (3.17)$$

These substitutions yield new equations of motion describing the displacement perturbations  $\delta u_i$  and stress perturbations  $\delta \sigma_{ij}$  as a function of new source terms  $\Delta f_i$  and  $\Delta T_{ij}$

$$\begin{aligned} \rho \frac{\partial^2 \delta u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \delta \sigma_{ij} &= \Delta f_i, \\ \delta \sigma_{ij} - c_{ijkl} \delta \epsilon_{kl} &= \Delta T_{ij}, \\ \delta \epsilon_{ij} &= \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) \\ &+ \text{perturbed initial and boundary conditions} \end{aligned} \quad (3.18)$$

The new source terms are

$$\Delta f_i = -\delta \rho \frac{\partial^2 u_i}{\partial t^2} \quad (3.19)$$

and

$$\Delta T_{ij} = \delta c_{ijkl} \epsilon_{kl}. \quad (3.20)$$

Two points are important to notice:

1. Eq.(3.18) states that every change of a material parameter acts as a source (Eq.(3.19) and Eq.(3.20)), but the perturbed wavefield is propagating in the unperturbed medium.
2. The new wave equation (3.18) has mathematically the same form as the unperturbed elastic wave equation, and hence its solution can be obtained in terms of Green's functions  $G_{ij}$  of the elastic wave equation.

The solution of the perturbed elastic equations of motion (3.18) in terms of the elastic Green's function  $G_{ij}(\mathbf{x}, t; \mathbf{x}', t')$  can be written as:

$$\begin{aligned} \delta u_i(\mathbf{x}, t) &= \int_V dV \int_0^T dt' G_{ij}(\mathbf{x}, t; \mathbf{x}', t') \Delta f_j(\mathbf{x}', t') \\ &- \int_V dV \int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(\mathbf{x}, t; \mathbf{x}', t') \Delta T_{jk}(\mathbf{x}', t'). \end{aligned} \quad (3.21)$$

Substituting the force and traction terms given by Eqs.(3.19) and (3.20) into Eq.(3.21) yields after some rearranging

$$\begin{aligned} \delta u_i(\mathbf{x}, t) &= - \int_V dV \int_0^T dt' G_{ij}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}', t') \delta \rho \\ &- \int_V dV \int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(\mathbf{x}, t; \mathbf{x}', t') \epsilon_{lm}(\mathbf{x}', t') \delta c_{ijklm} \end{aligned} \quad (3.22)$$

Utilization of Eq.(3.22) to solve the forward problem is known as Born approximation. In waveform tomography the Born approximation is not used to solve the forward problem. Instead the full elastic wave equation is solved. Equation (3.22) has the same form as the desired expression for the forward problem Eqs.(3.13):

$$\delta \mathbf{u} = \int_V dV \frac{\partial \mathbf{u}}{\partial \mathbf{m}} \delta \mathbf{m}. \quad (3.23)$$

Therefore the Frechét kernels  $\frac{\partial u_i}{\partial \mathbf{m}(\mathbf{x})}$  for the individual material parameters can be identified as:

$$\begin{aligned} \frac{\partial u_i}{\partial \rho} &= - \int_0^T dt' G_{ij}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}', t') \\ \frac{\partial u_i}{\partial c_{ijklm}} &= - \int_0^T dt' \frac{\partial G_{ij}}{\partial x'_k}(\mathbf{x}, t; \mathbf{x}', t') \epsilon_{lm}(\mathbf{x}', t') \end{aligned} \quad (3.24)$$

By definition the adjoint of the operator (3.23) can be written as

$$\delta \mathbf{m}'(\mathbf{x}) = \sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{\text{rec}}} \left[ \frac{\partial u_i}{\partial \mathbf{m}} \right]^* \delta u'_i(\mathbf{x}_\alpha, t'), \quad (3.25)$$

Because a linear operator and its transpose have the same kernels  $\partial u_i / \partial \mathbf{m}$ , the only difference arise in the variables of sum/integration, which are complementary. Inserting the integral kernels (3.24) in Eq.(3.25) yields

$$\begin{aligned} \delta \rho' &= - \sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{\text{rec}}} \int_0^T dt' G_{ij}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}, t) \delta u'_i(\mathbf{x}_\alpha, t'), \\ \delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \sum_{\alpha=1}^{N_{\text{rec}}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \epsilon_{lm}(\mathbf{x}, t) \delta u'_i(\mathbf{x}_\alpha, t'). \end{aligned}$$

The terms only depending on time  $t$  and the positions  $\mathbf{x}$  can be moved in front of the sum over the receivers

$$\begin{aligned} \delta \rho' &= - \sum_{\text{sources}} \int_0^T dt \frac{\partial^2 u_j}{\partial t^2}(\mathbf{x}, t) \sum_{\alpha=1}^{N_{\text{rec}}} \int_0^T dt' G_{ij}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u'_i(\mathbf{x}_\alpha, t'), \\ \delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm}(\mathbf{x}, t) \sum_{\alpha=1}^{N_{\text{rec}}} \int_0^T dt' \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u'_i(\mathbf{x}_\alpha, t'). \end{aligned} \quad (3.26)$$

Defining the wavefield

$$\Psi_j(\mathbf{x}, t) = \sum_{\alpha=1}^{N_{\text{rec}}} \int_0^T dt' G_{ij}(\mathbf{x}_\alpha, t'; \mathbf{x}, t) \delta u'_i(\mathbf{x}_\alpha, t'), \quad (3.27)$$

Eqs.(3.26) can be written as

$$\begin{aligned} \delta \rho' &= - \sum_{\text{sources}} \int_0^T dt \frac{\partial^2 u_j}{\partial t^2} \Psi_j, \\ \delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm} \frac{\partial \Psi_j}{\partial x_k}. \end{aligned} \quad (3.28)$$

The wavefield  $\Psi_j$  is generated by propagating the residual data  $\delta u'_i$  from the receiver positions backwards in time through the elastic medium. To obtain a more symmetric expression for the gradient  $\delta c'_{jklm}$  we exchange the indices  $j$  and  $k$  in Eqs. (3.21) - (3.28) and add both gradient expressions

$$\delta c'_{jklm} + \delta c'_{kjl m} = - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm} \left( \frac{\partial \Psi_j}{\partial x_k} + \frac{\partial \Psi_k}{\partial x_j} \right). \quad (3.29)$$

Because both gradients should be equal we get

$$\begin{aligned} 2\delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm} \left( \frac{\partial \Psi_j}{\partial x_k} + \frac{\partial \Psi_k}{\partial x_j} \right), \\ \delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm} \Theta_{jk}, \end{aligned} \quad (3.30)$$

with  $\Theta_{jk} = \frac{1}{2} \left( \frac{\partial \Psi_j}{\partial x_k} + \frac{\partial \Psi_k}{\partial x_j} \right)$ . Therefore the gradients for density and components of the elastic tensor are:

$$\begin{aligned} \delta \rho' &= - \sum_{\text{sources}} \int_0^T dt \frac{\partial^2 u_j}{\partial t^2} \Psi_j, \\ \delta c'_{jklm} &= - \sum_{\text{sources}} \int_0^T dt \epsilon_{lm} \Theta_{jk}, \end{aligned} \quad (3.31)$$

### 3.4 Estimation of an optimum step length $\mu_n$

The choice of the step length  $\mu_n$  in Eq. 3.10 is crucial for the convergence of the steepest gradient optimization method. I demonstrate this using a very familiar test problem for optimization routines, the Rosenbrock function with two unknown parameters (Rosenbrock [1960], Fig. 3.4)

$$f_r(x, y) = (1 - x)^2 + 100(y - x^2)^2 \quad (3.32)$$

The aim is to find the minimum of this function located at the point [1,1] which is surrounded by a very narrow valley. We start the search for the minimum at [-0.5,0.5]. An obvious first choice would be a constant step length. Fig. 3.4 (top) shows the convergence after 16000 iteration steps of the steepest descent method when choosing a step length  $\mu_n = 2e - 3$ . Note the large model update during the first iteration step, when the gradient of the Rosenbrock function is large. After reaching the narrow valley the gradient is much smaller and as a result the model updates are also decreasing. This leads to a very slow convergence speed. Especially near the minimum the model updates become very small. When choosing a larger step length ( $\mu_n = 2e - 3$ , Fig. 3.4 (bottom)) the model update is larger even when the gradient is small, but the code fails to converge at all. Instead it is trapped in a narrow part of the valley. To solve this problem a variable step length is introduced. For three test step lengths  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  three test models are calculated

$$\begin{aligned} \mathbf{m}_{\text{test1}} &= \mathbf{m}_n + \mu_1 \delta \mathbf{m}'_n \\ \mathbf{m}_{\text{test2}} &= \mathbf{m}_n + \mu_2 \delta \mathbf{m}'_n \\ \mathbf{m}_{\text{test3}} &= \mathbf{m}_n + \mu_3 \delta \mathbf{m}'_n \end{aligned} \quad (3.33)$$

and the corresponding L2-norms  $L2_1$ ,  $L2_2$  and  $L2_3$  are estimated (Fig. 3.5). The true misfit function (yellow line) can be approximated by fitting a parabola through the three points

$$L2_i = a\mu_i^2 + b\mu_i + c \quad (3.34)$$

where  $i \in \{1, 2, 3\}$  and  $a$ ,  $b$ ,  $c$  are the unknown coefficients. This system of equations can be written as matrix equation:

$$\begin{pmatrix} \mu_1^2 & \mu_1 & 1 \\ \mu_2^2 & \mu_2 & 1 \\ \mu_3^2 & \mu_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} L2_1 \\ L2_2 \\ L2_3 \end{pmatrix}$$

or

$$\mathbf{Ax} = \mathbf{b}. \quad (3.35)$$

The unknown coefficients of this matrix equation are formally defined by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}, \quad (3.36)$$

In the FWT code the solution vector  $\mathbf{x}$  is calculated by Gaussian elimination. In the following the step length at the extremum of the parabola defines the optimum step length  $\mu_{\text{opt}}$  (denoted as green square in Fig.3.5). This optimum step length is

$$\mu_{\text{opt}} = -\frac{b}{2a}. \quad (3.37)$$

The application of the variable step length calculation to the Rosenbrock test problem is shown in Fig. 3.6. The number of required iteration steps to reach the minimum is reduced by a factor 80 when compared with the constant step length gradient method. The only problem remaining is the slow convergence speed in the small valley of the Rosenbrock function, due to the fact that the update occurs along the gradient direction of the objective function resulting in a "criss-cross" pattern. This behaviour can be avoided by applying a nonlinear conjugate gradient method (chapter 3.5). In case of the FWT algorithm the three test step lengths for the individual material parameter classes are calculated by scaling the maximum of the gradient to the maximum of the actual models:

$$\begin{aligned} \mu_\lambda &= p \frac{\max(\lambda_n)}{\max(\delta\lambda_n)} \\ \mu_\mu &= p \frac{\max(\mu_n)}{\max(\delta\mu_n)} \\ \mu_\rho &= p \frac{\max(\rho_n)}{\max(\delta\rho_n)} s_{\text{rho}} \end{aligned} \quad (3.38)$$

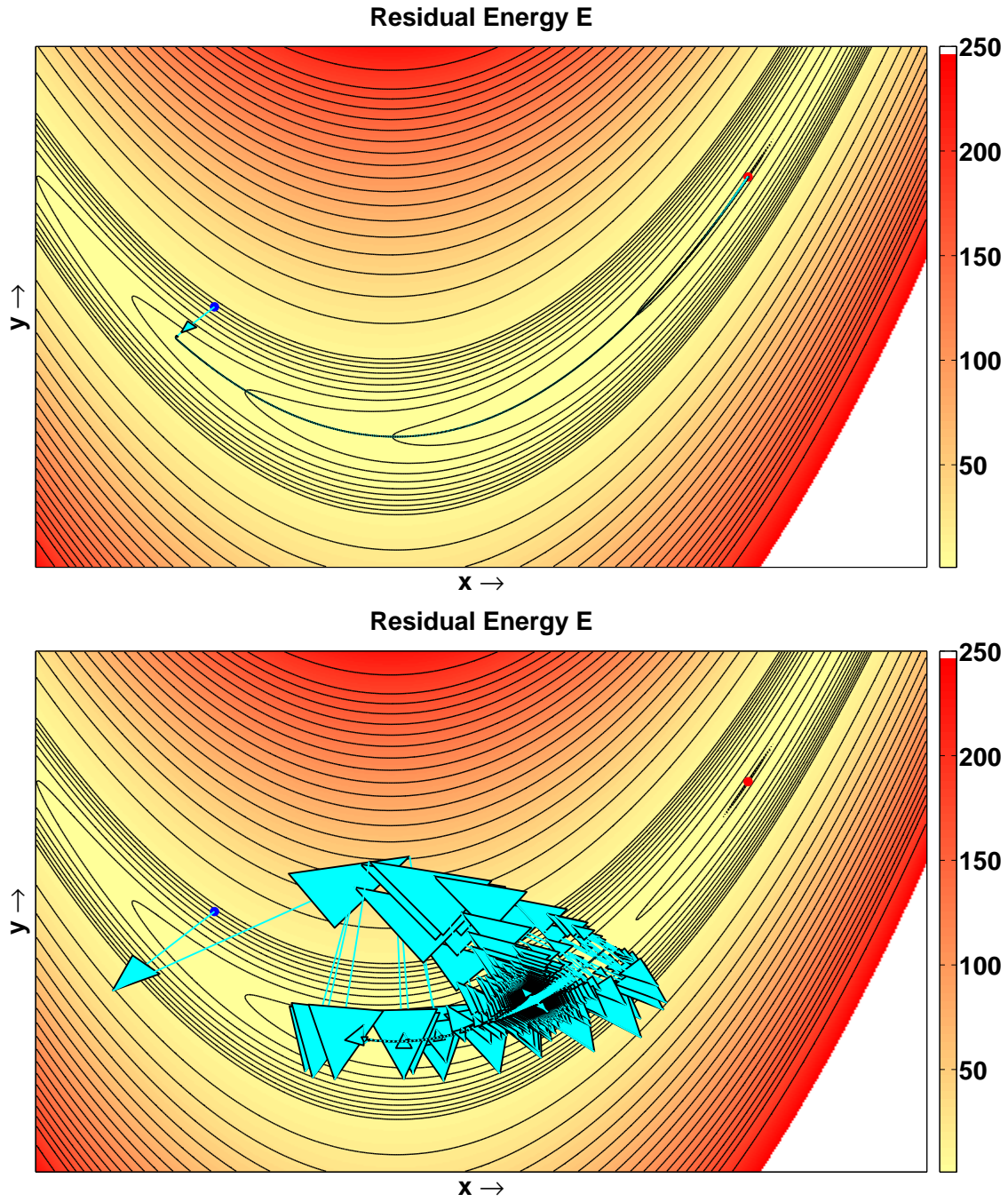


Figure 3.4: Results of the convergence test for the Rosenbrock function. The minimum is marked with a red point, the starting position with a blue point. The maximum number of iterations is 16000. The step length  $\mu_n$  varies between  $2e - 3$  (top) and  $6.1e - 3$  (bottom).

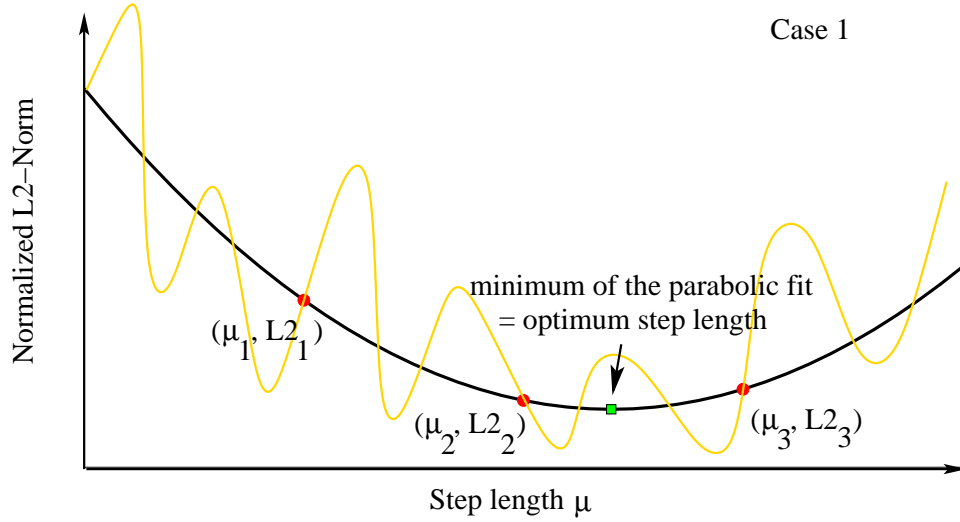


Figure 3.5: Line search algorithm to find the optimum step length  $\mu_{\text{opt}}$ : The true misfit function (yellow line) is approximated by a parabola fitting values of the objective function for 3 different step length.

Because changes of the density model are in most cases smaller than velocity changes the step length for the density update can be systematically reduced by a factor  $s_{\text{rho}}$ . All material parameters can be updated simultaneously or according to a hierarchical strategy. To save computational time the corresponding  $L_2$ -norms are calculated for a few representative shots. The number of shots depends on the complexity of the problem and the signal/noise ratio of the data. For the acoustic case the step length estimation by parabolic fitting works very well and leads to a smooth decrease of the misfit function during the FWT (Kurzmann (2007), personal communication, ?). For the multiparameter elastic FWT the misfit function consists of more local minima and therefore the decrease of the objective function is not as smooth as in the acoustic case. Brossier [2009] proposed a more intensive bracketing stage before applying the parabolic fit. Starting from  $p_1 = 0.0$  the test step lengths  $p_2$  and  $p_3$  are calculated to satisfy the following criteria:

$$\begin{aligned} L2_2(\mathbf{m}_{\text{test}2} = \mathbf{m}_n + \mu_2 \delta \mathbf{m}'_n) &< L2_1(\mathbf{m}_{\text{test}1} = \mathbf{m}_n) \\ L2_3(\mathbf{m}_{\text{test}3} = \mathbf{m}_n + \mu_3 \delta \mathbf{m}'_n) &> L2_2(\mathbf{m}_{\text{test}2} = \mathbf{m}_n + \mu_2 \delta \mathbf{m}'_n) \end{aligned} \quad (3.39)$$

This approach leads to a smoother decrease of the objective function, but also increases the number of required forward models.

### 3.5 Nonlinear Conjugate Gradient Method

To increase the convergence speed in narrow valleys it would be better to update the model at iteration step  $n$  not exactly along the gradient direction  $\delta \mathbf{m}_n$ , but along the conjugate direction  $\delta \mathbf{c}_n$

$$\delta \mathbf{c}_n = \delta \mathbf{m}_n + \beta_n \delta \mathbf{c}_{n-1}, \quad (3.40)$$

The first iteration step ( $n=1$ ) consists of a model update along the steepest descent direction:

$$\mathbf{m}_2 = \mathbf{m}_1 + \mu_1 \delta \mathbf{m}_1, \quad (3.41)$$

For all subsequent iteration steps ( $n > 1$ ) the model is updated along the conjugate direction:

$$\mathbf{m}_{n+1} = \mathbf{m}_n + \mu_n \delta \mathbf{c}_n, \quad (3.42)$$

where  $\delta \mathbf{c}_1 = \delta \mathbf{m}_1$ . The weighting factor  $\beta$  can be calculated in different ways [Nocedal and Wright, 2006]:

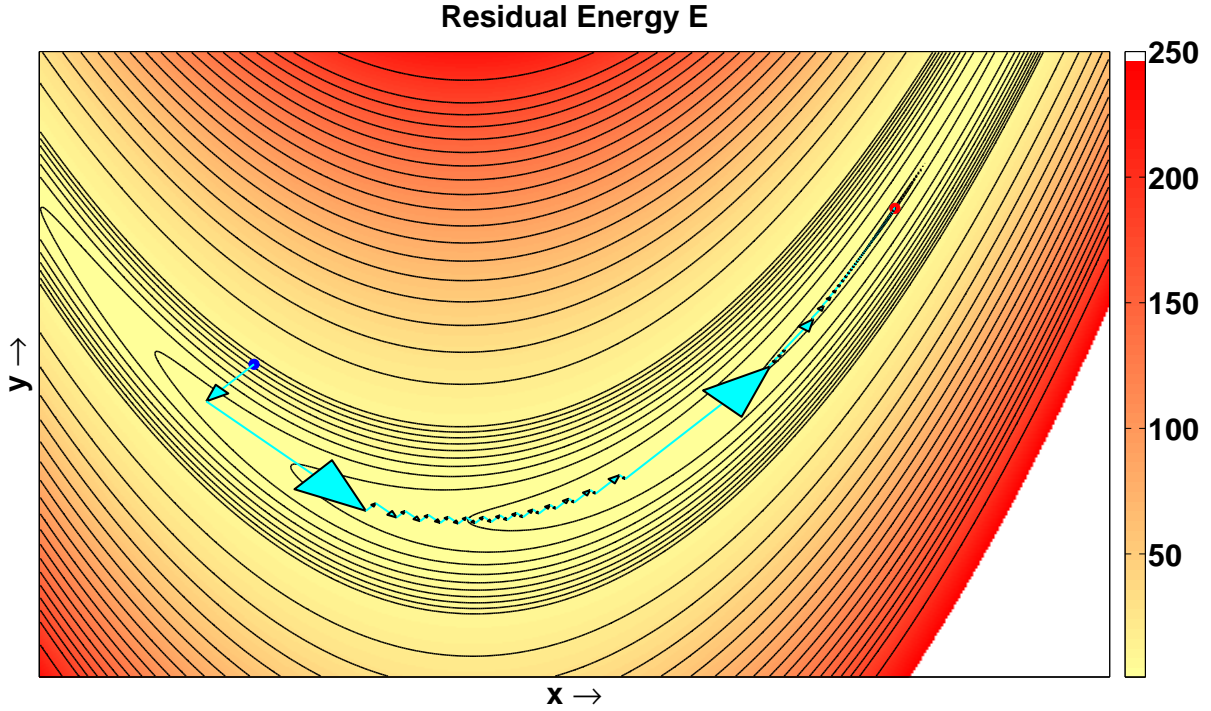


Figure 3.6: Results of the convergence test for the Rosenbrock function using the pure gradient method. The minimum is marked by a red point, the starting position by a blue point. The number of iterations is 200. The optimum step length is calculated at each iteration by the parabola fitting algorithm. Note the criss-cross pattern of the updates in the narrow valley and the slow convergence speed near the minimum.

1. Fletcher-Reeves [Fletcher and Reeves, 1964]:

$$\beta_n^{\text{FR}} = \frac{\delta \mathbf{m}_n^T \delta \mathbf{m}_n}{\delta \mathbf{m}_{n-1}^T \delta \mathbf{m}_{n-1}} \quad (3.43)$$

2. Polak-Ribière [Polak and Ribière, 1969]:

$$\beta_n^{\text{PR}} = \frac{\delta \mathbf{m}_n^T (\delta \mathbf{m}_n - \delta \mathbf{m}_{n-1})}{\delta \mathbf{m}_{n-1}^T \delta \mathbf{m}_{n-1}} \quad (3.44)$$

3. Hestenes-Stiefel [Hestenes and Stiefel, 1952]:

$$\beta_n^{\text{HS}} = \frac{\delta \mathbf{m}_n^T (\delta \mathbf{m}_n - \delta \mathbf{m}_{n-1})}{\delta \mathbf{c}_{n-1}^T (\delta \mathbf{m}_n - \delta \mathbf{m}_{n-1})} \quad (3.45)$$

We use the very popular choice  $\beta_n = \max[0, \beta_n^{\text{PR}}]$  which provides an automatic direction reset. This is important because subsequent search directions lose conjugacy requiring the search direction to be reset to the steepest descent direction. Note that the conjugate gradient method doesn't require any additional computational time because only the gradient  $\delta \mathbf{m}_n$  at two subsequent iterations has to be known. The application of the nonlinear conjugate gradient method combined with the variable step length calculation to the Rosenbrock function is shown in Fig. 3.7. The criss-cross pattern of the steepest descent method has vanished and the conjugate gradient method already converges after 30 iterations compared with 200 iteration steps of the pure gradient method.

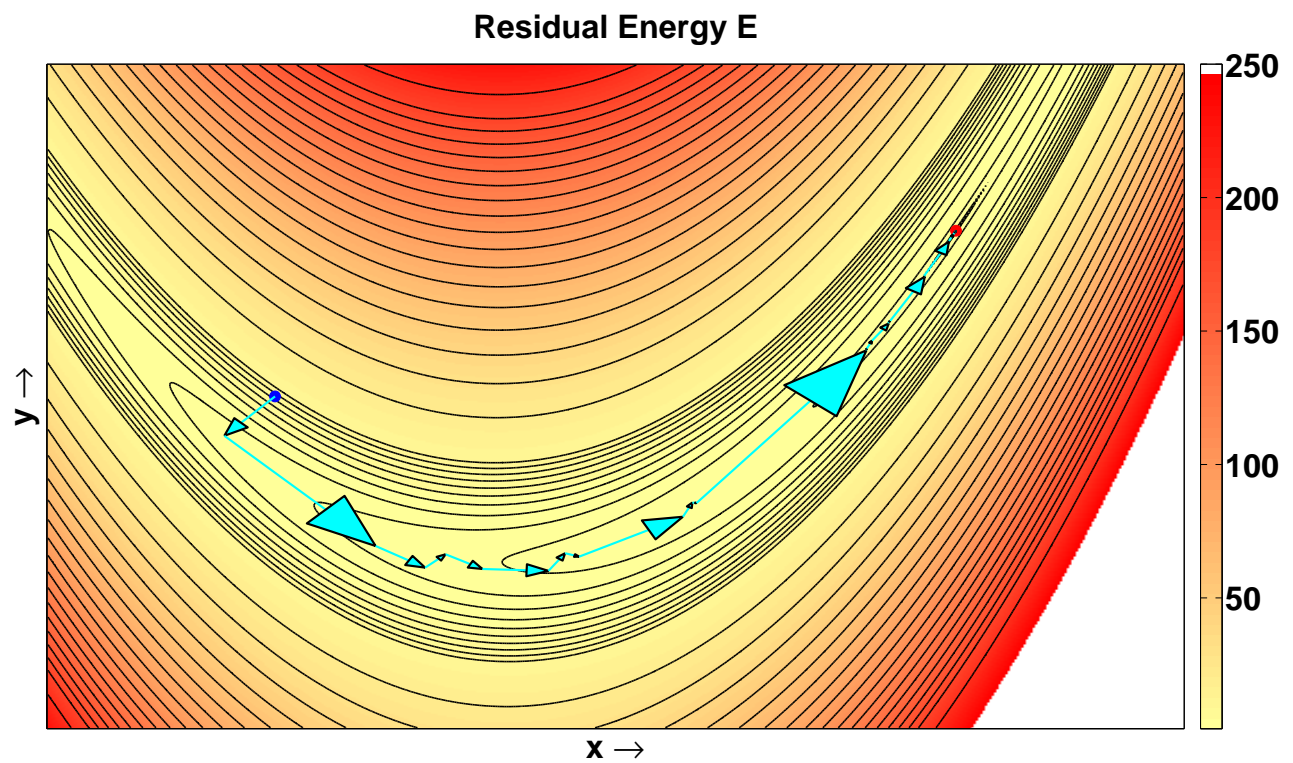


Figure 3.7: Results of the convergence test for the Rosenbrock function using the nonlinear conjugate gradient method, where the optimum step length is calculated with the parabolic fitting algorithm. The minimum is marked by a red cross, the starting point by a blue point. The number of iterations is 30.

### 3.6 The elastic FWT algorithm

In summary the FWT algorithm consists of the following steps:

1. Define a starting model  $\mathbf{m}_1$  in the parameter space. This model should represent the long wavelength part of the underground very well, because the FWT code is only capable to reconstruct structures at or below the dominant seismic wavelength due to its slow convergence speed, the nonlinearity of the problem and the inherent use of the Born approximation to calculate the gradient direction.
2. At iteration step  $n$  do:
  - (a) For each shot solve the forward problem, stated in Eq.(3.16) for the actual model  $\mathbf{m}_n$  to generate a synthetic dataset  $\mathbf{u}^{\text{mod}}$  and the wavefield  $\mathbf{u}(\mathbf{x}, t)$ .
  - (b) Calculate the residual seismograms  $\delta\mathbf{u} = \mathbf{u}^{\text{mod}} - \mathbf{u}^{\text{obs}}$  for the x- and y-components of the seismic data.
  - (c) Generate the wavefield  $\Psi(\mathbf{x}, t)$  by backpropagating the residuals from the receiver positions.
  - (d) Calculate the gradients  $\delta\mathbf{m}_n$  of each material parameter according to Eqs.(??).
  - (e) To increase the convergence speed an appropriate preconditioning operator  $P$  is applied to the gradient  $\delta\mathbf{m}$

$$\delta\mathbf{m}_n^p = P\delta\mathbf{m}_n \quad (3.46)$$

Examples of simple preconditioning operator are given in chapter ?? for a cross-well acquisition geometry and in chapter ?? for a reflection geometry.

- (f) For a further increase of the convergence speed calculate the conjugate gradient direction for iteration steps  $n \geq 2$ :

$$\delta\mathbf{c}_n = \delta\mathbf{m}_n^p + \beta\delta\mathbf{c}_{n-1}, \text{ with } \delta\mathbf{c}_1 = \delta\mathbf{m}_1^p \quad (3.47)$$

where the weighting factor

$$\beta^{\text{PR}} = \delta\mathbf{m}_n^p \frac{\delta\mathbf{m}_n^p - \delta\mathbf{m}_{n-1}^p}{\delta\mathbf{m}_{n-1}^p \delta\mathbf{m}_{n-1}^p} \quad (3.48)$$

by Polak-Ribière is used. The convergence of the Polak-Ribière method is guaranteed by choosing  $\beta = \max[\beta^{\text{PR}}, 0]$ .

- (g) Estimate the step length  $\mu_n$  by the line search algorithm described in chapter 3.4.
- (h) Update the material parameters using the gradient method

$$\mathbf{m}_{n+1} = \mathbf{m}_n - \mu_n \delta\mathbf{c}_n. \quad (3.49)$$

If the material parameters are not coupled by empirical relationships it is important to update all three elastic material parameters at the same time, otherwise strong artefacts may dominate the inversion result, especially in the case of very complex media.

3. If the residual energy  $E$  is smaller than a given value stop the iteration. Otherwise continue with the next iteration step.

## Chapter 4

# Source Wavelet Inversion

So far we assumed that only the material parameters are unknown, while the characteristics of the sources are perfectly known. For the application of FWI to a field dataset the source wavelet has to be estimated. In frequency domain the source wavelet  $s$  has the complex value

$$s = e + if$$

where  $i = \sqrt{-1}$ ,  $e$  and  $f$  are the real and imaginary parts, respectively. The seismograms of the vertical displacements of the modelled data can be described by:

$$v_r^m = (c_{v,r} + id_{v,r})(e + if)$$

where  $(c_{v,r} + id_{v,r})$  denotes the spike response of the vertical displacement and  $r$  the receiver location. Similar the seismograms of the vertical displacements of the field data are:

$$v_r^d = (a_{v,r} + ib_{v,r})$$

Update the real and imaginary parts of the source wavelet with the Newton method

$$\begin{aligned} \mathbf{e}_{n+1} &= \mathbf{e}_n - \mathbf{H}_n^{-1} \left( \frac{\partial E}{\partial e} \right)_n, \\ \mathbf{f}_{n+1} &= \mathbf{f}_n - \mathbf{H}_n^{-1} \left( \frac{\partial E}{\partial f} \right)_n. \end{aligned} \quad (4.1)$$

where  $\mathbf{H}$ ,  $\left( \frac{\partial E}{\partial e} \right)_n$  and  $\left( \frac{\partial E}{\partial f} \right)_n$  are the Hessian matrix and gradient vector for  $e$  and  $f$ , respectively. With the objective function

$$E = \frac{1}{2} \sum_r (v_r^m - v_r^d)(v_r^m - v_r^d)^* \quad (4.2)$$

the gradients and Hessian can be explicitly calculated

$$\begin{aligned} \frac{\partial E}{\partial e} &= \sum_r [e(c_{v,r}^2 + d_{v,r}^2) - a_{v,r}c_{v,r} - b_{v,r}d_{v,r}] \\ \frac{\partial E}{\partial f} &= \sum_r [f(c_{v,r}^2 + d_{v,r}^2) + a_{v,r}d_{v,r} - b_{v,r}c_{v,r}] \\ \frac{\partial^2 E}{\partial e^2} &= \sum_r (c_{v,r}^2 + d_{v,r}^2), \\ \frac{\partial^2 E}{\partial f \partial e} &= 0, \quad \frac{\partial^2 E}{\partial e \partial f} = 0, \\ \frac{\partial^2 E}{\partial f^2} &= \sum_r (c_{v,r}^2 + d_{v,r}^2). \end{aligned} \quad (4.3)$$

Inserting Eq. (4.3) in Eq. (4.1) leads to

$$\begin{aligned} \mathbf{e}_{n+1} &= \frac{\sum_r (a_{v,r} c_{v,r} - b_{v,r} d_{v,r})}{\sum_r (c_{v,r}^2 + d_{v,r}^2)} \\ \mathbf{f}_{n+1} &= -\frac{\sum_r (a_{v,r} d_{v,r} - b_{v,r} c_{v,r})}{\sum_r (c_{v,r}^2 + d_{v,r}^2)}. \end{aligned} \quad (4.4)$$

To stabilize the inversion a Marquardt-Levenberg regularization is required:

$$\begin{aligned} \mathbf{e}_{n+1} &= \mathbf{e}_n - (\mathbf{H}_n + \lambda_e \mathbf{I})^{-1} \left( \frac{\partial E}{\partial \mathbf{e}} \right)_n \\ \mathbf{f}_{n+1} &= \mathbf{f}_n - (\mathbf{H}_n + \lambda_f \mathbf{I})^{-1} \left( \frac{\partial E}{\partial \mathbf{f}} \right)_n \end{aligned}$$

where  $\lambda_e, \lambda_f$  are damping factors and  $\mathbf{I}$  the unity matrix. Therefore Eq. (4.4) can be written as

$$\begin{aligned} \mathbf{e}_{n+1} &= \frac{\sum_r (a_{v,r} c_{v,r} - b_{v,r} d_{v,r})}{\sum_r (c_{v,r}^2 + d_{v,r}^2) + \lambda_e} \\ \mathbf{f}_{n+1} &= -\frac{\sum_r (a_{v,r} d_{v,r} - b_{v,r} c_{v,r})}{\sum_r (c_{v,r}^2 + d_{v,r}^2) + \lambda_f}. \end{aligned} \quad (4.5)$$

The values of the damping factors can be expressed as fractions of the maximum value in the denominators

$$\lambda_e = \lambda_f = \epsilon_{stf} \max \left| \sum_r (c_{v,r}^2 + d_{v,r}^2) \right| \quad (4.6)$$

This approach is a stabilized Wiener deconvolution.

## Chapter 5

# Getting Started

In the following sections, we give a short description of the different modelling parameters, options and how the program is used in a parallel MPI environment.

### 5.1 Requirements

The parallelization employs functions of the Message Passing Interface (MPI). MPI has to be installed when compiling and running the SAVA software. At least two implementations exist for Unix-based networks: OpenMPI, MPICH2 and Intel-MPI. The LAM-MPI implementation is no longer supported by the developers. Currently all four implementation work with SAVA. OpenMPI, MPICH2 and Intel-MPI are MPI programming environments and development systems for heterogeneous computers on a network. As of the time of writing we get the best performance out of SAVA by using Intel-MPI together with the latest Intel-Compiler on a NEC-Linux Cluster. With MPI a dedicated cluster or an existing network computing infrastructure can act as a parallel computer. Fast network (infiniband) connections and RAM access are the most important issues for a good scaling of the SAVA code. The latest version of OpenMPI can be obtained from <http://www.open-mpi.org>. MPICH2 is available at <http://www-unix.mcs.anl.gov/mpi/mpich>. LAM-MPI can be downloaded here: <http://www.lam-mpi.org>, the commercial Intel-MPI from here: <https://software.intel.com/en-us/intel-mpi-library>.

#### 5.1.1 LAM

Even though outdated, LAM-MPI will be used to illustrate the setting up of the MPI implementation. In order to reproduce the following instructions, you must first install the LAM-MPI software. On SUSE LINUX systems the package can be installed with `yast`. The software can also be downloaded from <http://www.lam-mpi.org>. A good documentation of LAM/MPI is available at this site. Before you can run the SAVA software on more than one node you must boot LAM. This requires that you can connect to all of your nodes with `ssh` without a password. To enable `ssh` connection without a password you must generate authentication keys on each node using the command `ssh-keygen -t rsa`. These are saved in the file `~/.ssh/id_rsa.pub`. Copy all authentication keys into one file named `authorized_keys` and copy this file on all nodes into the directory `~/.ssh`.

Before you can boot LAM you must specify the IP addresses of the different processing elements in an ASCII file. An example file is `par/lamhosts`. Each line must contain one IP address. The first IP number corresponds to node 0, the second line to node 1 and so forth. Note that in older LAM-MPI implementations the `mpirun` command to run the FD programs must always be executed on node 0 !, i.e. you must log on node 0 (first line in the file `par/lamhosts`) and run the software on this node. In the last stable release of LAM-MPI, the node 0 just has to be listed in the `lamhosts` file (the order does not matter). To boot lam just do `lamboot -v par/lamhosts`. To run the software in serial on a single PC just type `lamboot` without specifying a `lamhosts` file. You can still specify more PEs in the FD software but all processes will be executed on the same CPU. Thus this only makes sense if you run the software on a multicore system. In this case, you should boot it without a `lamhosts` file and specify a total number of 2 processing elements (PEs). To shut down LAM again (before logout) use the command `lamclean -v`. However, it is not necessary to shut down/restart LAM after each logout/login.

### 5.1.2 How to run SAVA on the NEC-Linuxcluster at RZ Kiel

Before you can run SAVA on the Linux cluster at the computing centre in Kiel you have initialize Intel-MPI and Intel-compilers, and assure that the different nodes can communicate password-free. This has to be done only once.

1. Add the following lines to your `.bashrc` in your `$HOME` directory, to initialize Intel-MPI and the Intel-compiler:

```
. /opt/intel/composer_xe_2013_sp1/bin/compilervars.sh intel64
. /opt/intel/impi/4.1.1.036/intel64/bin/mpivars.sh
```

2. To setup a password-free communication between the different nodes generate a pair of authentication keys for ssh with:

```
[sungwXXX@nesh-fe2 ~]$ ssh-keygen -t dsa
```

You can accept the default values by hitting <return>.

3. Copy the file `$HOME/.ssh/id_dsa.pub` to `$HOME/.ssh/authorized_keys`.

Because SAVA can produce up to a few GB of data output, don't run the code from the home-directory. To submit a batch job it is required, that SAVA is located in the `$WORK` directory. Keep in mind, that the file system `$WORK` will not be automatically backedup, so do a manual backup from time to time. After compiling the code (see section 5.3), you can define and start a batch job with a shell script like this:

```
#!/bin/ksh
#PBS -l elapstim_req=96:00:00 # Walltime
#PBS -l cputim_job=1536:00:00 # akkumulated CPU-time per node
#PBS -l memsz_job=64gb       # RAM requirement
#PBS -b 4                    # number of nodes
#PBS -T mpich                # job type; mpich for Intel-MPI
#PBS -l cpunum_job=16        # number of CPUs per node
#PBS -N SAVA                 # name of the batch job
#PBS -o SAVA.out             # file for standard output
#PBS -j o                    # standard and
#PBS -q cllong               # batch class
#
#
. /opt/intel/composer_xe_2013_sp1/bin/compilervars.sh intel64
. /opt/intel/impi/4.1.1.036/intel64/bin/mpivars.sh
. /opt/modules/Modules/3.2.6/init/ksh
#
#
cd $WORK/SAVA/par
mpirun $NQSII_MPIOPTS -np 64 ../bin/denise SAVA.inp FWI_workflow.inp > denise.out
```

The individual parameters and possible batch-job classes are described in more detail on the homepage of the RZ Kiel [https://www.rz.uni-kiel.de/hpc/nec\\_cluster.html](https://www.rz.uni-kiel.de/hpc/nec_cluster.html) The most important parameters are

- *elapstim\_req*, which defines how long the job will actually run
- *cputim\_job* the accumulated CPU-time per node
- *memsz\_job* the required memory per node
- *-b* the number of nodes
- *mpich* the job type, in case of Intel-MPI you have to choose *mpich*
- *cpunum\_job* number of CPUs per node

- `-N` name of the batch job
- `-o` file name for standard output
- `-q` the requested batch-class.

An example for a job-file can be found in the `/SAVA/jobs` directory. The job can be submitted with

```
[sungwXXX@rzcluster ~]$ qsub SAVA.job
```

With

```
[sungwXXX@rzcluster ~]$ qstat
```

you can check the status of your Jobs and with

```
[sungwXXX@rzcluster ~]$ qdel <job_id>
```

you can cancel a submitted or running job, where `<job_id>` denotes the number at the first column of the status information, f.e.

```
[sungwXXX@nesh-fe2 jobs]$ qstat
```

RequestID	ReqName	UserName	Queue	Pri	STT	S	Memory	CPU	Elapse	R	H	M	Jobs
459654.ace-ssio	SAVA	sungwXXX	clmedium	0	RUN	-	13.89G	1077746.75	68303	Y	Y	Y	4
470371.ace-ssio	SAVA	sungwXXX	clmedium	0	QUE	-	0.00B	0.00	0	Y	Y	Y	4

```
[sungwXXX@nesh-fe2 jobs]$ qdel 459654
```

would kill the first job in the queue. For further information I again refer to the homepage of the RZ-Kiel: [https://www.rz.uni-kiel.de/hpc/nec\\_cluster.html](https://www.rz.uni-kiel.de/hpc/nec_cluster.html)

## 5.2 Installation

SAVA consists of four different packages:

- The source code
- A collection of benchmark models and pre-/postprocessing tools
- Matlab scripts for data pre-processing.
- The manuals for the different code versions.

Start with unpacking the source code package (e.g. by `tar -zxvf SAVA.tgz`) and changing to the directory SAVA (`cd SAVA`) you will find different subdirectories:

### **bin**

This directory contains all executable programs, generally SAVA and snapmerge. These executables are generated using the command `make <program>` (see below).

### **jobs**

This directory contains Batch-scripts to submit SAVA modelling/inversion runs on HPCs with PBS-batch system.

### **libseife**

This directory contains external contributions to SAVA for the implementation of a Butterworth frequency filter.

### **par**

Parameter files for SAVA modelling and inversion.

### **src**

This directory contains the complete source codes.

### 5.3 Compilation of SAVA

Before compiling SAVA you have to compile the additional libraries for timedomain filtering. In the SAVA/libcseife directory simply type:

```
-bash-2.05b$::~~/SAVA/libcseife> make
```

The source code of SAVA is located in the directory SAVA/src. To compile SAVA the name of the model function has to be entered in the MAKEFILE. Depending on your MPI environment (MPI distribution) you may need to modify the compiler options in src/Makefile. For a few typical platforms the compiler options are available in src/Makefile. It is often useful to enable a moderate level of optimization (typically -O3). The highest level of optimization -O4 can lead to a strong performance improvement. For example the optimization option -O4 of the hcc LAM compiler leads to a speedup of SAVA of approximately 30 per cent. Eventhough keep in mind that -O4 can also lead to crashes and compilation errors, when used in combination with certain compilers. No other changes in the Makefile should be necessary.

```
# Makefile for SAVA

#-----
# edit here:

# source code for model generation

MODEL_ANISO = model_aniso_triclin_new.c
EXEC= ../bin

# ON Linux cluster using OpenMPI
#CC=mpicc
#LFLAGS=-lm -lcseife
#CFLAGS=-O3
#SFLAGS=-L../libcseife
#IFLAGS=-I../libcseife

# ON Linux cluster using Intel-MPI
CC=mpiicc
LFLAGS=-lm -lcseife
CFLAGS=-O3
SFLAGS=-L../libcseife
IFLAGS=-I../libcseife

# On Desktop computer with LinuxMint 17, OpenMPI and gcc 4.8.2
#CC=mpicc
#LFLAGS=-lm -lcseife
#CFLAGS=-O3 -fno-stack-protector
#SFLAGS=-L../libcseife
#IFLAGS=-I../libcseife

# ALTIX
#CC=icc
#CFLAGS=-mp -O0 -ipo
#LFLAGS=-lmpi -lm -i-static

# after this line, no further editing should be necessary
# -----
```

To compile the program SAVA you must change to the src directory and execute:

```
-bash-2.05b$:/SAVA/src> make sava
```

The following (or a similar) output should occur:

```
[...]
mpiicc -O3 -c sources.c -I./../libcseife
mpiicc -O3 -c solvelin.c -I./../libcseife
mpiicc -O3 -c spat_filt.c -I./../libcseife
mpiicc -O3 -c splitsrc.c -I./../libcseife
mpiicc -O3 -c splitsrc_back.c -I./../libcseife
mpiicc -O3 -c splitrec.c -I./../libcseife
mpiicc -O3 -c stalta.c -I./../libcseife
mpiicc -O3 -c step_length_est.c -I./../libcseife
mpiicc -O3 -c step_length_est1.c -I./../libcseife
mpiicc -O3 -c stf.c -I./../libcseife
mpiicc -O3 -c taper.c -I./../libcseife
mpiicc -O3 -c taper_grad.c -I./../libcseife
mpiicc -O3 -c taper_grad_shot.c -I./../libcseife
mpiicc -O3 -c timedomain_filt.c -I./../libcseife
mpiicc -O3 -c time_window.c -I./../libcseife
mpiicc -O3 -c util.c -I./../libcseife
mpiicc -O3 -c wavelet.c -I./../libcseife
mpiicc -O3 -c wavelet_stf.c -I./../libcseife
mpiicc -O3 -c writemod.c -I./../libcseife
mpiicc -O3 -c write_par.c -I./../libcseife
mpiicc -O3 -c writedsk.c -I./../libcseife
mpiicc -O3 -c zero_fdveps.c -I./../libcseife
mpiicc -L./../libcseife sava.o calc_mat_change.o calc_mat_change_test.o
calc_res.o calc_opt_step.o calc_opt_step_test.o calc_energy.o catseis.o
checkfd_ssg_elastic.o conv_FD.o psource.o holbergcoeff.o comm_ini.o
exchange_v.o exchange_s.o exchange_L2.o fft.o fft_filt.o forward_mod.o
[...] snap_ssg.o seismo_ssg.o surface_elastic_2nd.o writemod.o
write_par.o writedsk.o zero_fdveps.o -o ../bin/sava -lm -lcseife
```

The program snapmerge that is used to merge the snapshots (see below) can be compiled with "make snapmerge". Since this is not a MPI program (no MPI functions are called) the MPI libraries are not required and any standard compiler (like gcc and cc) can be used to compile this program. The executables sava and snapmerge are located the directory bin.

## 5.4 Running the program

Each SAVA run reads the required parameters from the parameter files par/SAVA.inp and par/FWI\_workflow.inp. A detailed description of the parameters can be found in chapter 6. The command to start a simulation on 8 processor with the lowest priority of -19 (in order to allow working on the a workstation while running a simulation) is as follows. Please note, that we assume you have navigated to the folder SAVA/par and all parameter files are located in this directory.

```
mpirun -np 8 nice -19 ../bin/sava SAVA.inp FWI_workflow.inp
```

If you use LAMMPI the command `lamboot -v lamhost` must be executed on node 0 which is the PE where `./par/lamhosts` is the file containing IP addresses of all computing nodes. It is often useful to save the standard output of the program for later reference. The screen output may be saved to SAVA.out using

```
mpirun -np 8 nice -19 ../bin/sava SAVA.inp FWI_workflow.inp > SAVA.out
```

After the output of geometry and model parameters the code starts the time stepping and displaying timing information:

```
=====
MYID=0 ***** Starting simulation (forward model) for shot 1 of 1 *****
=====
```

```
Number of samples (nts) in source file: 3462
Number of samples (nts) in source file: 3462
Message from function wavelet written by PE 0
1 source positions located in subdomain of PE 0
have been assigned with a source signal.
```

```
PE 0 outputs source time function in SU format to start/source_signal.0.su.shot1
```

```
Computing timestep 1000 of 3462
```

```
**Message from update_v (printed by PE 0):
Updating particle velocities ... finished (real time: 0.00 s).
particle velocity exchange between PEs ... finished (real time: 0.00 s).
```

```
**Message from update_s (printed by PE 0):
Updating stress components ... finished (real time: 0.00 s).
stress exchange between PEs ... finished (real time: 0.00 s).
total real time for timestep 1000 : 0.01 s.
```

```
Computing timestep 2000 of 3462
```

```
**Message from update_v (printed by PE 0):
Updating particle velocities ... finished (real time: 0.00 s).
particle velocity exchange between PEs ... finished (real time: 0.00 s).
```

```
**Message from update_s (printed by PE 0):
Updating stress components ... finished (real time: 0.00 s).
stress exchange between PEs ... finished (real time: 0.00 s).
total real time for timestep 2000 : 0.01 s.
```

```
Computing timestep 3000 of 3462
```

```
**Message from update_v (printed by PE 0):
Updating particle velocities ... finished (real time: 0.00 s).
particle velocity exchange between PEs ... finished (real time: 0.00 s).
```

```
**Message from update_s (printed by PE 0):
Updating stress components ... finished (real time: 0.00 s).
stress exchange between PEs ... finished (real time: 0.00 s).
total real time for timestep 3000 : 0.01 s.
```

```
PE 0 is writing 11 seismograms (vx) to
    su/SAVA_US_x.su.shot1.it1
PE 0 is writing 11 seismograms (vy) to
    su/SAVA_US_y.su.shot1.it1
```

```
**Info from main (written by PE 0):
CPU time of program per PE: 17 seconds.
Total real time of program: 18.08 seconds.
```

```

Average times for
velocity update:      0.003 seconds
stress update:        0.002 seconds
velocity exchange:    0.000 seconds
stress exchange:      0.000 seconds
timestep:             0.005 seconds

```

## 5.5 Postprocessing

The wavefield snapshots can be merged using the program *snapmerge*. The program *snapmerge* is not a MPI program. Therefore, it can be executed without MPI and the *mpirun* command. You can run *snapmerge* on any PC since a MPI environment (e.g. LAM) is not required. You may therefore copy the snapshot outputs of the different nodes to another non-MPI computer to merge the files together. *snapmerge* reads the required information from the SAVA parameter file. Simply type

```
-bash-2.05b$~/SAVA/par> ../bin/snapmerge SAVA.inp
```

Depending on the model size the merge process may take a few seconds or hours. The output should read like this:

```

pressure (files: ./snap/test.bin.p.??? ).

writing merged snapshot file to  ./snap/test.bin.p
Opening snapshot files: ./snap/test.bin.p.???  ... finished.
Copying... ... finished.
Use
xmovie n1=100 n2=100 < ./snap/test.bin.p loop=1 labell=Y label2=X title=%g
to play movie.

```

XX Add description of visualization with Paraview here. XX

## Chapter 6

# Definition of parameters for the modelling and inversion code

The geometry of the FD grid and all parameters for the wavefield simulation and inversion have to be defined in a parameter file (which we name in this case *SAVA/par/SAVA.inp*). Parameters changing during the waveform inversion are defined in a separate file (which we name in this case *SAVA/par/FWI\_workflow.inp*). This allows the flexible combination of different inversion parameters and therefore an implementation of complex FWI workflows. In the following we will explain every input parameter section in detail.

### 6.1 Input file with fixed parameters SAVA.inp

Most lines in the parameter file are organized as follows:

```
description_of_parameter_(VARNAME)_(switches) = parameter value
```

where VARNAME denotes the name of the global variable in which the value is saved in all functions of the program. The possible values are described in switches. A comment line is indicated by a # on the very first position of a line. The meaning of the different parameters is described in the following.

### Domain decomposition

```
#----- Domain Decomposition -----  
number_of_processors_in_x-direction_(NPROCX) = 2  
number_of_processors_in_y-direction_(NPROCY) = 2  
number_of_processors_in_z-direction_(NPROCZ) = 2  
#
```

Parallelization is based on domain decomposition (see Figure 6.1), i.e. each processing element (PE) updates the wavefield within his portion of the grid. The model is decomposed by the program into sub grids. After decomposition each processing elements (PE) saves only his sub-volume of the grid. NPROCX, NPROCY and NPROCZ specify the number of processors in x-, y-, z-direction, respectively (Figure 6.1). The total number of processors thus is  $NP=NPROCX*NPROCY*NPROCZ$ . This value must be specified when starting the program with the mpirun command: *mpirun -np <NP> ../bin/SAVA SAVA.inp FWI\_workflow.inp* (see section 5.4). If the total number of processors in SAVA.inp and the command line differ, the program will terminate immediately with a corresponding error message. Obviously, the total number of PEs ( $NPROCX*NPROCY*NPROCZ$ ) used to decompose the model should be less equal than the total number of CPUs which are available on your parallel machine. If you use LAM and decompose your model in more domains than CPUs are available two or more domains will be updated on the same CPU (the program will not terminate and will produce the correct results). However, this is only efficient if more than one processor is available on each node. In order to reduce the amount of data that needs to be exchanged between PEs, you should decompose the model into more or less cubic sub grids. In our example, we

use 2 PEs in each direction: NPROCX=NPROCY=NPROCZ=2. The total number of PEs used by the program is NPROC=NPROCX\*NPROCY\*NPROCZ=8.

## Spatial discretization

```
#----- 2-D Grid -----
number_of_gridpoints_in_x-direction_(NX) = 200
number_of_gridpoints_in_y-direction_(NY) = 200
number_of_gridpoints_in_z-direction_(NZ) = 200
#
file_for_grid_spacing_in_x-direction_(in_m)_(DX_FILE) = ./model/dx.dat
file_for_grid_spacing_in_y-direction_(in_m)_(DY_FILE) = ./model/dy.dat
file_for_grid_spacing_in_z-direction_(in_m)_(DZ_FILE) = ./model/dz.dat
#
```

These lines specify the size of the total numerical grid (Figure 6.1). NX, NY and NZ give the number of grid points in the x-, y- and z-direction, respectively. The grid spacing in all three spatial directions can be variable, therefore the spatial discretization in x-, y- and z-direction is defined in the files dx.dat, dy.dat and dz.dat, respectively. If you simply want to use a constant spatial grid spacing enter only this value in the files. A variable grid in e.g. in x-direction can be defined by the following dx.dat file

```
50 10.0
50 5.0
100 2.5
```

which means the first 50 gridpoints in x-direction have a spacing of 10.0 m, the next 50 gridpoints a spacing of 5.0 m and the final 100 gridpoints a spacing of 2.5 m. Obviously the sum of all gridpoints in this example should be NX=200. To allow for a consistent domain decomposition NX/NPROCX, NY/NPROCY and NZ/NPROCZ must be integer values. To avoid numerical dispersion the wavefield must be discretized with a certain number of gridpoints per wavelength. The number of gridpoints per wavelength required, depends on the order of the spatial FD operators used in the simulation (see section 2.3.1). In the current FD software, 2nd order operators are implemented. The criterion to avoid numerical dispersion reads:

$$DH \leq \frac{v_{min}}{2f_c n} \quad (6.1)$$

where  $\frac{v_{min}}{2f_c}$  is the smallest wavelength propagating through the model.  $v_{min}$  denotes the minimum phase velocity in the model, and  $f_c$  is the center frequency of the source wavelet. The program assumes that the maximum frequency of the source signal is approximately two times the center frequency. The value of n for different FD operators is tabulated in table 2.2. The criterion 6.1 is checked by the FD software. If the criterion is violated a warning message will be displayed in the SAVA output section “— CHECK FOR GRID DISPERSION —”. Please note, that the FD-code will NOT terminate due to grid dispersion, only a warning is given in the output file.

## Time stepping

```
#-----Time Stepping -----
time_of_wave_propagation_(in_sec)_(TIME) = 1.8e-4
timestep_(in_seconds)_(DT) = 5.2e-8
#
```

The propagation time of seismic waves in the entire model is TIME. The time stepping interval (DT) has to fulfill the stability criterion (2.27) in section 2.3.2. The program checks these criteria for the entire model, outputs a warning message if these are violated, stops the program and will output the time step interval for a stable model run.

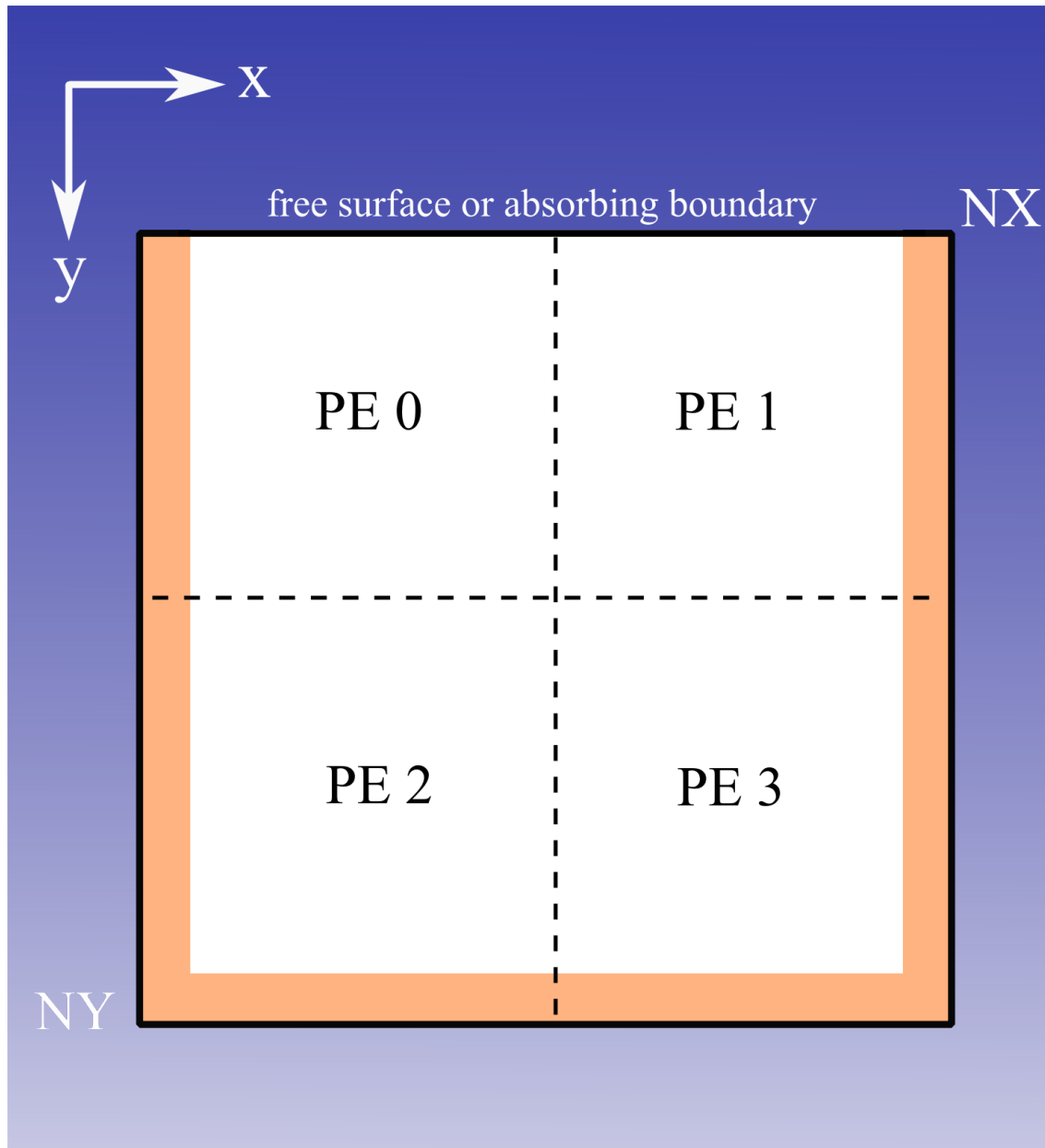


Figure 6.1: Geometry of the numerical FD grid using 2 processors in x-direction ( $NPROCX=2$ ), 2 processors in y-direction ( $NPROCY=2$ ) and 2 processors in z-direction ( $NPROCZ=2$ ). Each processing element (PE) is updating the wavefield in its domain. At the top of the numerical mesh the PEs apply a free surface boundary condition, otherwise an absorbing boundary condition (PML). The size of the total grid is NX grid points in x-direction, NY gridpoints in y-direction and NZ gridpoints in z-direction. The size of each sub-grid thus is  $NX/NPROCX \times NY/NPROCY \times NZ/NPROCZ$  gridpoints. XX Add 3D example XX

## Sources

```
#-----Source-----
# Shape_of_source-signal:
(ricker=1;fumue=2;SIN*3=3;from_ASCII_FILE=4;from_BIN_FILE=5)_(QUELLART) = 1
SIGNAL_FILE = ./model/signal.bin
# shot coordinates, amplitude and center frequency are defined in:
SOURCE_FILE = ./model/source
#
```

3 built-in wavelets of the seismic source are available. The corresponding time functions are defined in `src/wavelet.c`. You may modify the time functions in this file and recompile to include your own analytical wavelet or to modify the shape of the built-in wavelets.

### Ricker wavelet

$$r(\tau) = (1 - 2\tau^2) \exp(-\tau^2) \quad \text{with} \quad \tau = \frac{\pi(t - 1.5/f_c - t_d)}{1.0/f_c} \quad (6.2)$$

### Fuchs-Müller wavelet

$$f_m(t) = \sin(2\pi(t - t_d)f_c) - 0.5 \sin(4\pi(t - t_d)f_c) \quad \text{if} \quad t \in [t_d, t_d + 1/f_c] \quad \text{else} \quad f_m(t) = 0 \quad (6.3)$$

### $\sin^3$ wavelet

$$s3(t) = 0.75\pi f_c \sin(\pi(t + t_d)f_c)^3 \quad \text{if} \quad t \in [t_d, t_d + 1/f_c] \quad \text{else} \quad s3(t) = 0 \quad (6.4)$$

In these equations,  $t$  denotes time and  $f_c$  is the center frequency.  $t_d$  is a time delay which can be defined for each source position in `SOURCE_FILE`. Note that the symmetric (zero phase) Ricker signal is always delayed by  $1.0/f_c$ , which means that after one period the maximum amplitude is excited at the source location. These 5 source wavelets and the corresponding amplitude spectra for a center frequency of  $f_c = 50$  Hz and a delay of  $t_d = 0$  are plotted in Figure 6.2. Note the delay of the Ricker signal described above. The Fuchs-Müller wavelet has a slightly higher center frequency than the Ricker wavelet and covers a broader frequency range.

You may also use your own time function as the source wavelet (for instance the signal of the first arrival recorded by a geophone at near offsets). Specify `QUELLART=3` and save the samples of your source wavelet in ASCII-format in `SIGNAL_FILE`. `SIGNAL_FILE` should contain one sample per line. It should thus look like:

```
0.0
0.01
0.03
...
```

The time interval between the samples must equal the time step interval (DT) of the FD simulation (see above) ! Therefore it might be necessary to resample/interpolate a given source time function with a smaller sample rate. You may use the matlab script `mfiles/resamp.m` to resample your external source signal to the required sampling interval.

The location of the sources and other characteristics have to be defined in an external ASCII file (`SOURCE_FILE`) that has the following format:

```
NSRC
%  XSRC YSRC ZSRC TD FC AMP    SOURCE_TYPE (NSRC lines)
```

In the following lines, you can define certain parameters for each source point: the first line must be the overall number of sources (NSRC). XSRC is the x-coordinate of a source point (in meter), YSRC is the y-coordinate of a source point (in meter) and ZSRC is the z-coordinate of a source point (in meter). TD is the excitation time (time-delay) for the source point [in seconds], FC is the center frequency of the source signal [in Hz], and AMP is the maximum amplitude of the source signal.

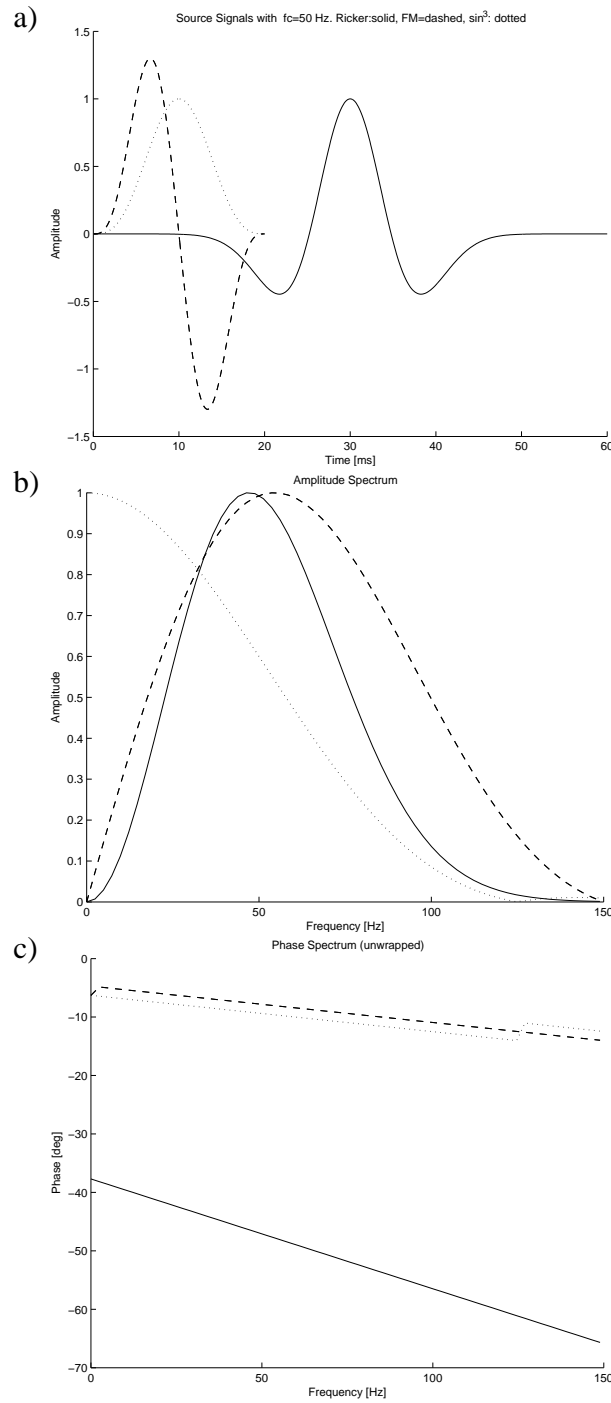


Figure 6.2: Plot of built-in source wavelets (equations 6.2, 6.3, 6.4) for a center frequency of  $f_c = 50$  Hz ( $TS = 1/f_c = 0.02$ s): Ricker signal (solid), Fuchs-Müller signal (dashed),  $\sin^3$ -signal (dotted). a) Time function, b) amplitude spectrum, c) phase spectrum.

The value of SOURCE\_TYPE defines:

- SOURCE\_TYPE = 1: explosive source
- SOURCE\_TYPE = 2: force in x-direction
- SOURCE\_TYPE = 3: force in y-direction
- SOURCE\_TYPE = 4: force in z-direction
- SOURCE\_TYPE = 5: double couple in x-y-plane
- SOURCE\_TYPE = 6: double couple in y-z-plane
- SOURCE\_TYPE = 7: double couple in x-z-plane

The SOURCE\_FILE = ./sources/source.dat that defines an explosive source at  $x_s = 2592.0$  m,  $y_s = 2106.0$  m and  $y_s = 5304.0$  m with a center frequency of 5 Hz (no time delay) is

```
2592.0      2106.0      5304.0      0.0      5.0      1.0      1
```

## Model input

```
#----- Model -----
read_model_from_MFILE(yes=1) (READMOD) = 0
MFILE = model/test
#
```

If READMOD=1, the components of the stiffness tensor and density model grids are read from external binary files. MFILE defines the basic file name that is expanded by the following extensions: c\_1111: ".c1111", c\_1112: ".c1112", ..., c\_3333: ".c3333" and density model: ".rho". In the example above, the model files thus are: "model/test.c1111", "model/test.c1112", ..., "model/test.c3333" and "model/test.rho".

In these files, each material parameter value must be saved as 32 bit (4 byte) native float. Components of the elastic tensor must be in Pa, density values in  $kg/m^3$ . For more details see src/readmod\_aniso.c. The number of samples for the entire model in the x-direction is NX, the number of values in the y-direction is NY and the number of values in the z-direction is NZ. The file size of each model file thus must be  $NX*NY*NZ*4$  bytes. You may check the model structure using the SU command ximage: xmovie n1=<NZ> n2=<NX> < model/test.vp .

If READMOD=0 the model is generated "on the fly" by SAVA, i.e. it is generated internally before the time loop starts. If READMOD=0 this function is called in SAVA.c and therefore must be specified in src/Makefile (at the top of src/Makefile, see section 5.3). If you change this file, for example to change the model structure, you need to re-compile SAVA by changing to the src directory and "make sava".

## Q-approximation

```
#-----Q-approximation-----
Number_of_relaxation_mechanisms_(L) = 0
L_Relaxation_frequencies_(FL) = 5.0
Tau_value_for_entire_model_(TAU0) = 0.00001
#
```

These lines are a relict from the isotropic viscoelastic code. We have to think how to handle viscoelasticity for the triclinic anisotropic case ...

## Boundary conditions

```
#-----Boundary conditions-----
# left boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_LE) = 1
boundary_width_(in_gridpoints)_(FWLE) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPLE) = 8.0
if_PML_(N_SIGMA_LE,SIGMA_LE) = 2.0 , 15000.0
if_PML_(N_KAPPA_LE,KAPPA_LE) = 1.0 , 1.0
if_PML_(N_ALPHA_LE,ALPHA_LE) = 1.0 , 0.0
#
# right boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_RI) = 1
boundary_width_(in_gridpoints)_(FWRI) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPRI) = 8.0
if_PML_(N_SIGMA_RI,SIGMA_RI) = 2.0 , 15000.0
if_PML_(N_KAPPA_RI,KAPPA_RI) = 1.0 , 1.0
if_PML_(N_ALPHA_RI,ALPHA_RI) = 1.0 , 0.0
#
# back boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_BA) = 1
boundary_width_(in_gridpoints)_(FWBA) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPBA) = 8.0
if_PML_(N_SIGMA_BA,SIGMA_BA) = 2.0 , 15000.0
if_PML_(N_KAPPA_BA,KAPPA_BA) = 1.0 , 1.0
if_PML_(N_ALPHA_BA,ALPHA_BA) = 1.0 , 0.0
#
# front boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_FR) = 1
boundary_width_(in_gridpoints)_(FWFR) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPFR) = 8.0
if_PML_(N_SIGMA_FR,SIGMA_FR) = 2.0 , 15000.0
if_PML_(N_KAPPA_FR,KAPPA_FR) = 1.0 , 1.0
if_PML_(N_ALPHA_FR,ALPHA_FR) = 1.0 , 0.0
#
# top boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_TO) = 1
boundary_width_(in_gridpoints)_(FWTO) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPTO) = 8.0
if_PML_(N_SIGMA_TO,SIGMA_TO) = 2.0 , 15000.0
if_PML_(N_KAPPA_TO,KAPPA_TO) = 1.0 , 1.0
if_PML_(N_ALPHA_TO,ALPHA_TO) = 1.0 , 0.0
#
# bottom boundary
type_of_absorbing_boundary_(FREE_SURFACE=0,PML=1,AB=2,PERIODIC=3)_(ABTYPE_BO) = 1
boundary_width_(in_gridpoints)_(FWBO) = 20
if_AB_percentage_of_amplitude_decay_at_outer_edge_(DAMPBO) = 8.0
if_PML_(N_SIGMA_BO,SIGMA_BO) = 2.0 , 15000.0
if_PML_(N_KAPPA_BO,KAPPA_BO) = 1.0 , 1.0
if_PML_(N_ALPHA_BO,ALPHA_BO) = 1.0 , 0.0
#
```

The boundary conditions are applied on each side face of the Cartesian modelling grid can be define separately, left ABTYPE\_LE, right ABTYPE\_RI, back ABTYPE\_BA, front ABTYPE\_FR, top ABTYPE\_TO and bottom AB-

TYPE\_BO. If ABTYPE\_XX = 0 a free surface boundary condition is applied, while ABTYPE\_XX = 1 and ABTYPE\_XX = 2 applies perfectly matched layer (PML) or a simple but less effective absorbing boundary condition [Cerjan et al., 1985], respectively. Note that the absorbing frames are always located INSIDE the model space, i.e. parts of the model structure are covered by the absorbing frame, in which no physically meaningful wavefield propagates. You should therefore consider the frame width when you design the model structure and the acquisition geometry (shot and receivers should certainly be placed outside). The parameters FWXX defines the thickness of the absorbing boundary condition at each side of the model. N\_SIGMA\_XX, SIGMA\_XX, N\_KAPPA\_XX, KAPPA\_XX and N\_ALPHA\_XX, ALPHA\_XX define the damping profile within the PML boundary layer and DAMPXX the amount of damping within the Cerjan absorbing boundary conditions. The parameters have to adapt to your modelling/inversion problem.

In some cases, it is useful to apply periodic boundary conditions (see section 2.2.3). If ABTYPE\_XX=3 no absorbing boundaries are installed at the respective and opposite side of the grid. Instead, wavefield information is copied from one side of the grid to the other and vice versa. Therefore a wave which leaves the model on one side enters the model again at the opposite side and vice versa.

## Wavefield snapshots

```
#-----Snapshots-----
output_of_snapshots_(SNAP) (yes>0) = 0
# SNAP=0:      no snapshots
# SNAP=SNAP+1: add particle-velocities
# SNAP=SNAP+2: add pressure (hydrophones)
# SNAP=SNAP+4: add curl and div
# SNAP=SNAP+8: add particle-accelerations
# SNAP=SNAP+16: add stress components
first_snapshot_(in_sec)_(TSNAP1) = 0.02
last_snapshot_(in_sec)_(TSNAP2) = 2.0
increment_(in_sec)_(TSNAPINC) = 0.02
min_x_position_(in_m)_(XSNAPMIN) = 0.0
max_x_position_(in_m)_(XSNAPMAX) = 2000.0
increment_y-direction_(IDX) = 1
min_y_position_(in_m)_(YSNAPMIN) = 1000.0
max_y_position_(in_m)_(YSNAPMAX) = 1010.0
increment_y-direction_(IDY) = 1
min_z_position_(in_m)_(ZSNAPMIN) = 0.0
max_z_position_(in_m)_(ZSNAPMAX) = 2000.0
increment_z-direction_(IDZ) = 1
data-format_(SNAP_FORMAT) (ASCII(2);BINARY(3)) = 3
basic_filename_(SNAP_FILE) = ./snap/fd3d_aniso
#
```

If SNAP > 0, wavefield information (particle velocities, pressure, curl and divergence of particle velocities, particle-accelerations or stress components) for the entire model is saved on the hard disk (assure that enough free space is on disk!). Each PE is writing his sub-volume to disk. The filenames have the basic filename SNAP\_FILE plus an extension that indicates the PE number in the logical processor array (see Figure 6.1), i.e. the PE with number PEno writes his wavefield to SNAPFILE.PEno. The output format can be either ASCII (SNAP\_FORMAT=2) or BINARY (SNAP\_FORMAT=3). The first snapshot is written at TSNAP1 seconds of seismic wave traveltimes to the output files, the second at TSNAP1+TSNAPINC seconds etc. The last snapshots contains wavefield at TSNAP2 seconds. Make sure that TSNAP1 is larger or equal than the FD-timestep DT, otherwise no output will be produced. Note that the file sizes increase during the simulation. The snapshot files might become quite LARGE. It may therefore be necessary to reduce the amount of snapshot data by increasing IDX, IDY, IDZ and/or TSNAPINC. Furthermore only parts of the 3D model can be written to disk, where the minimum and maximum spatial coordinates in each spatial dimension is defined by XSNAPMIN, XSNAPMAX, YSNAPMIN, YSNAPMAX, ZSNAPMIN and ZSNAPMAX, respectively. In order to merge the separate snapshot of each PE after the completion of the wave modelling, you can use the program

snapmerge (see Chapter 5.2, section **src**). The bash command line to merge the snapshot files can look like this: `../bin/snapmerge SAVA.inp`. A description how to visualize the 3D wavefield with Paraview can be found in section 5.5.

## Receiver

```
#-----Receiver-----
read_receiver_positions_from_file_(yes=1)_(READREC) = 1
REC_FILE = ./model/receiver.dat
reference_point_for_receiver_coordinate_system_(REFREC) = 0.0 , 0.0, 0.0
# if READREC=1 the following three lines are ignored
position_of_first_receiver_(in_m)_(XREC1,YREC1,ZREC1) = 1.0 , 10.0 , 5.0
position_of_last_receiver_(in_m)_(XREC2,YREC2,ZREC2) = 3.0 , 10.0 , 5.0
distance_between_two_adjacent_receivers_(in_gridpoints)_(DXREC,DYREC,DZREC) = 100 , 100, 100
#
```

The locations of the receivers on the FD grid may either be specified in a separate file `REC_FILE` or in this parameter file. If `READREC=1` receiver locations are read from the ASCII-file `REC_FILE`. Each line contains the x-, y- and z-coordinates of one receiver. To give an example of a receiver file, the following 3 lines specify 3 receivers located at constant depth (2106.0 m). However, the receiver coordinates change in x-direction (starting at 540 m) and therefore lining up along the x-axis, while the y-coordinate is also constant (500.0 m)

```
540.0 500.0 2106.0
1080.0 500.0 2106.0
1620.0 500.0 2106.0
```

These receiver coordinates in `REC_FILE` are shifted by `REFREC[1]`, `REFREC[2]`, `REFREC[3]` in the x-, y- and z-direction, respectively. This allows for completely moving the receiver spread without modifying `REC_FILE`. This may be useful for the simulation of moving profiles in reflection seismics.

If `READREC=0` the receiver locations must be specified in the parameter file. In this case, it is assumed that the receivers are located along a straight line. The first receiver position is defined by `(XREC1, YREC1, ZREC1)`, and the last receiver position by `(XREC2, YREC2, ZREC2)` (see Figure 6.1). The spacing between receivers in the three spatial directions are `DXREC`, `DYREC` and `DZREC` grid points.

Receivers are always located on full grid indices, i.e. a receiver that is located between two grid points will be shifted by the FD program to the closest next grid point. It is not yet possible to output seismograms for arbitrary receiver locations since this would require a certain wavefield interpolation.

**It is important to note that the actual receiver positions defined in `REC_FILE` or in `SAVA.inp` may vary by  $DX/2$  and/or  $DY/2$  and/or  $DZ/2$  due to the staggered positions of the particle velocities and stress tensor components.**

## Seismograms

```
#----- Seismograms -----
output_of_seismograms_(SEISMO) = 1
# SEISMO=0:          no seismograms
# SEISMO=SEISMO+1:   add particle-velocities
# SEISMO=SEISMO+2:   add pressure (hydrophones)
# SEISMO=SEISMO+4:   add curl and div
# SEISMO=SEISMO+8:   add particle-accelerations
# SEISMO=SEISMO+16:  add stress components
samplingrate_(in_timesteps!)(NDT) = 1
data-format_(SU(1);ASCII(2);BINARY(3)) = 1
basic_filename_(SEIS_FILE) = ./su/fd3d_aniso
#
```

If `SEISMO>0` seismograms recorded at the receiver positions are written to the corresponding output files `SEIS_FILE`. Depending on `SEISMO` the seismograms consist of the x- and y-component of particle velocity, pressure, curl and divergence of the particle velocity, particle accelerations or stress components recorded at the receiver locations.

The curl and divergence of the particle velocities are useful to separate between P- and S-waves in the snapshots of the wavefield. SAVA calculates the divergence and magnitude of the curl of the particle velocity field according to Dougherty and Stephen [1988]. The motivation for this is as follows. According to Morse and Feshbach Morse and Feshbach [1953] the energy of P- and S-wave particle velocities is, respectively,

$$E_p = (\lambda + 2\mu) (\text{div}(\vec{v}))^2 \quad \text{and} \quad E_s = \mu |\text{rot}(\vec{v})|^2 \quad . \quad (6.5)$$

$\lambda$  and  $\mu$  are the Lamè parameters, and  $\vec{v}$  is the particle velocity vector. The sampling rate of the seismograms is `NDT*DT` seconds. In case of a small time step interval and a high number of time steps, it might be useful to choose a high `NDT` in order to avoid a unnecessary detailed sampling of the seismograms and consequently large files of seismogram data. Keep in mind though that the application of FWI requires `NDT=1`. Possible output formats of the seismograms are SU, ASCII and BINARY. It is recommended to use SU format for saving the seismograms. The main advantage of this format is that the time step interval (`NDT*DT`) and the acquisition geometry (shot and receiver locations) are stored in the corresponding SU header words. Also additional header words like offset are set by SAVA. This format thus facilitates a further visualization and processing of the synthetic seismograms. Note, however, that SU cannot handle sampling rates smaller than 1.0e-6 seconds and the number of samples is limited to about 32.000. In such cases, you should increase the sampling rate by increasing `NDT`. If this is impossible (for example because the Nyquist criterion is violated) you must choose a different output format (ASCII or binary).

## Monitoring the simulation

```
# each PE is printing log-information to LOG_FILE.MYID
log-file_for_information_about_progress_of_program_(LOG_FILE) = ./log/fd3d_aniso.log
info_of_processing_element_zero_to_stdout_(yes=1/no=0)_(LOG) = 1
```

SAVA can output a lot of useful information about the modelling parameters and the status of the modelling process etc. The major part of this information is output by PE 0. If `LOG=1`, PE 0 writes this info to stdout, i.e. on the screen of your shell. This is generally recommended to monitor the modelling process. You may want to save this screen info to an output file by adding `> SAVA.out` or `&tee SAVA.out`. to your starting command. If `LOG=1` all other processes with PE number (`PEno`) greater than zero will write their information to `LOG_FILE.PEno`. If you specify `LOG=2` PE 0 will also output information to `LOG_FILE.0`. As a consequence only little information is written directly to the screen of your shell. On supercomputers where you submit modelling jobs to a queuing system as batch jobs `LOG=2` may be advantageous. In case of `LOG=2`, you may still watch the simulation by checking the content of `LOG_FILE.0` for example by using the Unix commands `more` or `tail`. After finishing the program the timing information is written to the ASCII file `log/test.log.0.timings`. This feature is useful to benchmark your local PC cluster or supercomputer. If `LOG=0` no output from node 0 will be written, neither to stdout nor to an LOG file. There will be also no output of timing information to the ASCII file `log/test.log.0.timings`.

## Checkpointing

```
#-----Checkpoints -----
read_wavefield_from_checkpoint_file_(yes=1/no=0)_(CHECKPTREAD) = 0
save_wavefield_to_checkpoint_file_(yes=1/no=0)_(CHECKPTWRITE) = 0
checkpoint_file_(CHECKPTFILE) = tmp/checkpoint_SAVA
```

These options are obsolete and not be supported in the current version of SAVA.

## General inversion parameters

```
#----- General SAVA inversion parameters -----#
number_of_TDFWI_iterations_(ITERMAX) = 10
output_of_jacobian_(JACOBIAN) = gradient/SAVA
seismograms_of_measured_data_(DATA_DIR) = su/INC/fd3d_aniso
forward_modelling_only_(yes=0)_FWI_(yes=1)_(INVMAT) = 1
point_source_backpropagation_(all_comp=1/x_comp=2/y_comp=3/z_comp=4)_(QUELLTYPB) = 1
#
```

This section covers some general inversion parameters. The maximum number of iterations are defined by ITERMAX. The switch INVMAT controls if only the forward modelling code should be used (INVMAT=0), e.g. to calculate synthetic seismograms or a complete FWI run (INVMAT=1). In case of INVMAT=0 the parameters in the workflow file (section 6.2) are ignored, but a workflow file still has to be defined. The seismic sections of the real field data are located in the DATA\_DIR. Depending on the recorded field data different components of the seismic sections can be backpropagated. For three component data (x-, y- and z-component) set QUELLTYPB=1, only x-component (QUELLTYPB=2), only y-component (QUELLTYPB=3) or only z-component (QUELLTYPB=4).

## Optimization method

```
# ----- Optimization-Method ----- #
gradient_method_(PCG=1/LBFGS=2)_(GRAD_METHOD) = 2
#
```

During the FWI the misfit function can be minimized by different optimization methods. Currently a preconditioned conjugate gradient (PCG) and the quasi-Newton method L-BFGS method (see f.e. Nocedal and Wright [2006]) can be used.

## Reduce inversion grid

```
#----- Reduce inversion grid -----#
use_only_every_DTINV_time_sample_for_gradient_calculation_(DTINV) = 3
#
```

To reduce the memory requirements during an inversion one can define that only every DTINV time sample is used for the calculation of the gradients. To set this parameter appropriately one has to keep in mind the Nyquist criterion to avoid aliasing effects.

## Step length estimation

```
#----- Step length estimation -----#
maximum_model_change_of_maximum_model_value_(EPS_SCALE) = 0.01
maximum_number_of_attempts_to_find_a_step_length_(STEPMAX) = 4
SCALEFAC = 2.0
#
```

For the step length estimation a parabolic line search method proposed by Sourbier et al. [2009a,b], Brossier [2009] and Nocedal and Wright [2006] is implemented. For this step length estimation further test forward modellings are needed. The vector L2t contains the misfit values and the vector epst contains the corresponding step length. During the forward modelling of an iteration step the misfit norm of the data residuals is calculated. The value L2t[1] then contains the misfit from the forward modelling and the corresponding epst[1] value is 0.0.

The step lengths for the different parameters are defined as:  
 $\text{EPSILON} = \text{EPS\_SCALE} * \text{m\_max/grad\_max}$   
 $\text{EPSILON} = \text{epst}[i] * \text{m\_max/grad\_max}$

where  $m\_max$  is the maximum value of the corresponding model parameter in the whole model and  $grad\_max$  is the maximum absolute value of the gradient.

For a better definition of the parabola the improved line search is now trying to estimate a steplength  $epst[2]$  with  $L2t[2] < L2t[1]$ . If the code is not able to find an appropriate steplength using the user-defined value  $EPS\_SCALE$  (f.e.  $EPS\_SCALE = 0.01 = 1\%$  change in terms of  $m\_max/grad\_max$ ), the code divides this steplength by the variable  $SCALEFAC$  and calculates the misfit norm again. If this search fails after  $STEPMAX$  attempts SAVA exits with an error message. If the algorithm has found an appropriate value for  $epst[2]$ , it is trying to estimate a steplength  $epst[3]$  with  $L2t[3] > L2t[2]$ , by increasing the steplength

$EPS\_SCALE += EPS\_SCALE/SCALEFAC$ .

If a corresponding value  $epst[3]$  can be found after  $STEPMAX$  forward modellings, SAVA can fit a parabola through the 3 points  $(L2t[i], epst[i])$  and estimates an optimum step length at the minimum of the parabola. If the  $L2$ -value  $L2t[3]$  after  $STEPMAX$  forward models is still smaller than  $L2t[2]$  the optimum steplength estimated by parabolic fitting will be not larger than  $epst[3]$ .

## Trace killing

```
#----- Trace killing -----#
apply_trace_killing_(yes=1)_(TRKILL) = 0
TRKILL_FILE = ./trace_kill/trace_kill.dat
```

To mute noisy or unwanted traces during FWI, the parameter  $TRKILL$  is introduced. If  $TRKILL$  is set to 1, all traces defined in the parameter file  $TRKILL\_FILE$  are muted. The file should include a mute table, where the rows have the same lengths as the number of receivers and the columns reflects the number of sources. A 1 (ONE) indicates a mute of the trace, while a 0 (ZERO) means that this trace should NOT be muted.

## Time damping

```
#----- Time windowing and damping -----#
files_with_picked_times_(PICKS_FILE) = ./picked_times/picks_
```

If time damping of the seismograms is activated in the workflow file by setting  $TIMWIN!=0$  picked times of specific seismic phases, like first arrivals, for each source and receiver must be specified in a separate file. The folder and file name can be set with the parameter  $PICKS\_FILE$ . The files must be named like this  $[PICKS\_FILE]_{sourcenum}.dat$ . So the number of sources in  $(SRCREC)$  must be equal to the number of files. Each file must contain the picked times for every receiver. Other important parameters are set in the workflow file (see section 6.2).

## Name of the misfit log file

```
#----- MISFIT LOG FILE -----#
log_file_for_misfit_evolution_(MISFIT_LOG_FILE) = LOG_TEST.dat
#
```

The name of the misfit log file can be changed with the parameter  $MISFIT\_LOG\_FILE$ . The columns of the misfit log file contain information about the step length and misfit function values acquired during the step length estimation and the stage number  $nstage$ :

```
opteps epst[1] epst[2] epst[3] L2t[1] L2t[2] L2t[3] L2t[1] nstage
```

When a frequency filter is applied information about the corner frequencies are also written to the misfit log file.

```
opteps epst[1] epst[2] epst[3] L2t[1] L2t[2] L2t[3] L2t[1] FC_low FC_high nstage
```

## Time-lapse FWI mode

```
# ----- FWT double-difference time-lapse mode ----- #
activate_time_lapse_mode_(yes=1)_(TIMELAPSE) = 0
# if TIMELAPSE == 1, DATA_DIR should be the directory containing the data differences
# between time t0 and t1
seismograms_of_synthetic_data_at_t0_(DATA_DIR_T0) = su/CAES_spike_time_0/SAVA_CAES
#
```

If TIMELAPSE=1 the spatial FWI is replaced by a double-difference time-lapse FWI Denli and Huang [2009], al Hagrey et al. [2014]. In this case DATA\_DIR defines the data differences between the baseline data at time t0 and the time-lapse data at t1. For existing SU-files the data differences can be calculated with the shell script *time\_lapse\_data\_diff.sh* in the par-directory. The location of the baseline data can be defined with DATA\_DIR\_T0.

## Elastic Reverse Time Migration

```
# ----- Elastic Reverse Time Migration ----- #
apply_reverse_time_migration_(yes=1)_(RTM) = 0
#
```

If RTM=1 an elastic 3D Reverse Time Migration is applied for the field data defined in the directory DATA\_DIR. If time-lapse mode is activated (TIMELAPSE=1) the time-lapse data will be migrated. The workflow file (section 6.2) should only contain one stage. The resulting migrated seismic sections are written to the directory JACOBIAN. Currently Reverse Time Migration is only defined for the L2-Norm (L2NORM=2).

## 6.2 Workflow file with variable inversion parameters FWI\_workflow.inp

Complex FWI workflows can be designed with the input file shown in table 6.1. Each line represents a FWI stage with a specific combination of different inversion parameters, defined in the columns.

### Abort criterion

Beside the parameter ITERMAX a second abort criterion is implemented in SAVA which is using the relative misfit change within the last two iterations. The relative misfit of the current iteration step and the misfit of the second to last iteration step is calculated with regard to the misfit of the second to last iteration step. If this relative change is smaller than PRO the inversion aborts or proceeds to the next inversion stage.

### Frequency filtering

To tame the nonlinearity of the inversion problem Butterworth frequency filters can be applied to the source wavelet and field data.

- TIME\_FILT=1 applies a lowpass frequency filter with an upper corner frequency FC\_high.
- TIME\_FILT=2 applies a bandpass frequency filter with a lower corner frequency FC\_low and upper corner frequency FC\_high.

The order of the Butterworth filter is defined by the parameter ORDER.

### Time damping

Multiple or complex reflections can significantly increase the nonlinearity of the inverse problem. Different time-damping strategies are implemented in SAVA to

- TIMEWIN=1 reads traveltimes picks of the first arrival from the PICKS\_FILES defined in the parameter file (section 6). A constant time-delay TWIN+ can be applied to each pick.
- TIMEWIN=2 applies a time-damping from a constant time TWIN+ for all receivers and shots.

The amount of damping can be defined by the parameter GAMMA.

### Spatial filtering of gradients

To suppress short wavelength artefacts below the source and receiver positions the gradients can be smoothed.

- SPATFILTER=1 applies a wavenumber domain damping with a Gaussian function

$$\hat{g}(k_x, k_y) = g(k_x, k_y) \exp(-WD\_DAMP(k_x^2 + k_y^2))$$

to the gradients  $g(k_x, k_y)$ . The amount of damping can be controlled by the parameter WD\_DAMP.

- SPATFILTER=2 applies a damped least squares technique to the gradients. The size of the filter is defined by WD\_DAMP.

### Preconditioning

To accelerate the convergence speed of the optimization method and avoid the convergence in a local minimum, amplitude loss with depth due to geometrical spreading and reflections in the upper model parts have to be compensated. In case of Quasi-Newton or Full-Newton methods these effects are corrected by the inverse Hessian. For (conjugate) gradient methods different approximations of the inverse Hessian can be used as preconditioning operator.

- EPRECOND=1 approximates the inverse of the Hessian by the absolute value of the forward wavefield [Shin et al., 2001]:

$$H_a^{-1} = \left\{ \int dt |\mathbf{u}(\mathbf{x}_s, \mathbf{x}, t)|^2 \right\}^{-1}.$$

- EPRECOND=3 approximates the inverse of the Hessian by a zero-lag correlation of the absolute value of the forward wavefield with an approximation of the receiver Greens function contribution [Plessix and Mulder, 2004]:

$$H_a^{-1} = \left\{ \int dt |\mathbf{u}(\mathbf{x}_s, \mathbf{x}, t)|^2 \left[ \operatorname{asinh}\left(\frac{x_r^{\max} - x}{z}\right) - \operatorname{asinh}\left(\frac{x_r^{\min} - x}{z}\right) \right] \right\}^{-1},$$

where  $x_r^{\min}$ ,  $x_r^{\max}$ ,  $\mathbf{x}_s$  denote the minimum, maximum receiver and source positions.

### Misfit definition

Different objective functions can have a significant impact on the nonlinearity of the inverse problem. Changing the misfit function between the modelled data  $\mathbf{u}$  and field data  $\mathbf{d}$  does only change the backpropagated residuals in the FWI algorithm.

- LNORM=2 sets the misfit function to the "classical" L2 norm of the data residuals Eq. (3.3)

$$E_{L2} = \frac{1}{2} \sum_i^{ns} \sum_j^{nr} (\mathbf{u}_{ij} - \mathbf{d}_{ij})^2.$$

In this case the misfit is scaled with the energy of the measured seismograms.

- LNORM=5 sets the misfit function to the global correlation norm [Choi and Alkhalifah, 2012]

$$E_{GC} = - \sum_i^{ns} \sum_j^{nr} \left[ \frac{\mathbf{u}_{ij}}{\|\mathbf{u}_{ij}\|} \cdot \frac{\mathbf{d}_{ij}}{\|\mathbf{d}_{ij}\|} \right]$$

### Source wavelet inversion

As discussed in chapter 4 the estimation of the source wavelet is vital for a successful FWI. With STF\_INV=1 a source wavelet inversion by a stabilized Wiener deconvolution is activated. This requires one additional forward model run per shot to estimate the Greens function solution for the actual model. The parameter NTR\_STF defines how many traces in the vicinity of the shot point are used. In case of dispersive wavefields it is recommended to limit the source wavelet inversion only to the near-offset traces to avoid the interpretation of model parameter changes as source wavelet. In case of NTR\_STF<0 all traces are incorporated for the wavelet inversion. EPS\_STF denotes the regularization parameter defined in Eq. (4.6). So far the source wavelet will be only estimated from the vertical component data.

### Offset-Windowing

In some cases the application of an offset-window can be useful to achieve a "layer-stripping" update of the model parameters from top to bottom.

- OFFSET\_MUTE=1 mutes all traces with an offset larger than OFFSETC (far-offset mute).
- OFFSET\_MUTE=2 mutes all traces with an offset smaller than OFFSETC (near-offset mute).

### Density model update restriction

Because changes of the density model are in most cases smaller than velocity changes the step length for the density update can be systematically reduced by a factor SCALERHO (see Eq. (3.38)).

PRO	TIME_FILT	FC_low	FC_high	ORDER	TIME_WIN	GAMMA	TWIN-	TWIN+	SPATFILTER	WD_DAMP	EPRECOND	LNORM	STF_INV	NTR_STF	EPS_STF	OFFSET_MUTE	OFFSETC	SCALERHO
0.01	1	0.0	1.7	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	1.7	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	3.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	5.15	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	1.7	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	1.7	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	3.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	5.15	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	1.7	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	1.75	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	1.7	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	2.9	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	2.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	3.55	0	160	3	2	0	-4	1e-1	0	10.0	0.0
0.01	1	0.0	9.0	6	2	1e2	0.0	5.15	0	160	3	2	0	-4	1e-1	0	10.0	0.0

Table 6.1: Example of a complex hierachical multiparameter FWI workflow definition (modified after Kurzmann [2012]). The colors represent different parameter groups.

## **Chapter 7**

### **Example 1 - coming soon ...**

# Bibliography

- S. al Hagrey, D. Köhn, and W. Rabbel. Geophysical assessments of renewable gas energy compressed in geologic pore storage reservoirs. *SpringerPlus*, 3(1):1–16, 2014.
- A. Brenders and R. Pratt. Full waveform tomography for lithospheric imaging: results from a blind test in a realistic crustal model. *Geophys. J. Int.*, 168:133–151, 2007.
- R. Brossier. *Imagerie sismique à deux dimensions des milieux visco-élastiques par inversion des formes d’ondes : développements méthodologiques et applications*. PhD thesis, Université de Nice - Sophia Antipolis, 2009.
- C. Cerjan, D. Kosloff, R. Kosloff, and M. Reshef. A nonreflecting boundary condition for discrete acoustic and elastic wave equations. *Geophysics*, 50:705–708, 1985.
- Y. Choi and T. Alkhalifah. Application of multi-source waveform inversion to marine streamer data using the global correlation norm. *Geophysical Prospecting*, 60:748–758, 2012.
- Y. Choi, D. Min, and C. Shin. Frequency-domain elastic full waveform inversion using the new pseudo-hessian matrix: Experience of elastic marmousi-2 synthetic data. *Bull., Seis Soc. Am.*, 98:2402–2415, 2008a.
- Y. Choi, D. Min, and C. Shin. Two-dimensional waveform inversion of multi-component data in acoustic-elastic coupled media. *Geophysical Prospecting*, 56:863–881, 2008b.
- R. Courant, K. Friedrichs, and H. Lewy. Über die partiellen Differenzengleichungen der mathematischen Physik. *Mathematische Annalen*, 100:32–74, 1928.
- R. Courant, K. Friedrichs, and H. Lewy. On the partial difference equations of mathematical physics. *IBM Journal*, pages 215–234, March 1967.
- E. Crase. High-order (space and time) finite-difference modeling of the elastic wave equation. In *60th Annual International Meeting, Society of Exploration Geophysicists, Expanded Abstracts*, pages 987–991, 1990.
- H. Denli and L. Huang. Double-difference elastic waveform tomography in the time domain. In *SEG Technical Program Expanded Abstracts*, pages 2302–2306, 2009.
- M. Dougherty and R. Stephen. Seismic energy partitioning and scattering in laterally heterogeneous ocean crust. *Pure Appl. Geophys.*, 128(1/2):195 – 239, 1988.
- R. Fletcher and C. M. Reeves. Function minimization by conjugate gradients. *The Computer Journal*, 7(2):149–154, 1964.
- O. Gauthier, J. Virieux, and A. Tarantola. Two-dimensional nonlinear inversion of seismic waveforms - numerical results. *Geophysics*, 51(7):1387–1403, 1986.
- M. Hestenes and E. Stiefel. Methods of conjugate gradients for solving linear systems. *Journal of Research of the National Bureau of Standards*, 49(6):409–436, 1952.
- O. Holberg. Computational aspects of the choice of operator and sampling interval for numerical differentiation in large-scale simulation of wave phenomena. *Geophysical Prospecting*, 35:629–655, 1987.

- H. Igel, P. Mora, and B. Riollot. Anisotropic wave propagation through finite-difference grids. *Geophysics*, 60: 1203–1216, 1995.
- C. Jastram. *Seismische Modellierung mit Finiten Differenzen höherer Ordnung auf einem Gitter mit vertikal variierendem Gitterabstand*. PhD thesis, Universität Hamburg, 1992.
- D. Komatitsch and R. Martin. An unsplit convolutional perfectly matched layer improved at grazing incidence for the seismic wave equation. 72(5):155 – 167, 2007.
- A. Kurzmann. *Applications of 2D and 3D full waveform tomography in acoustic and viscoacoustic complex media*. PhD thesis, Karlsruhe Institute of Technology (KIT), 2012. available at <http://nbn-resolving.de/urn:nbn:de:swb:90-344211>.
- A. Levander. Fourth-order finite-difference P-SV seismograms. *Geophysics*, 53(11):1425–1436, 1988.
- P. Moczo, J. Kristek, and L. Halada. *The Finite-Difference Method for Seismologists. An Introduction*. Comenius University, Bratislava, 2004.
- P. Mora. Nonlinear two-dimensional elastic inversion of multioffset seismic data. *Geophysics*, 52:1211 – 1228, 1987.
- P. Morse and H. Feshbach. *Methods of theoretical physics*. McGraw-Hill Book Company, New York, 1953.
- J. Nocedal and S. Wright. *Numerical Optimization*. Springer, New York, 2006.
- A. Pica, J. Diet, and A. Tarantola. Nonlinear inversion of seismic reflection data in a laterally invariant medium. 55 (3):284–282, 1990.
- R.-E. Plessix and W. A. Mulder. Frequency-domain finite-difference amplitude-preserving migration. *Geophysical Journal International*, 157(3):975–987, 2004.
- E. Polak and G. Ribière. Note sur la convergence de méthodes de directions conjuguées. *Revue Francaise d'Informatique et de Recherche Opérationnelle*, 16:35–43, 1969.
- R. Pratt. Inverse theory applied to multi-source cross-hole tomography. Part II: Elastic wave-equation method. *Geophysical Prospecting*, 38:311–329, 1990.
- R. Pratt. Seismic waveform inversion in the frequency domain, Part 1: Theory and verification in a physical scale model. *Geophysics*, 64:888–901, 1999.
- R. Pratt. Velocity models from frequency-domain waveform tomography: Past, present and future. In *66th EAGE conference and exhibition, Expanded Abstracts*, pages 181–182, Paris, France, 2004.
- R. Pratt and M. Worthington. Inverse theory applied to multi-source cross-hole tomography. Part I: Acoustic wave equation method. *Geophysical Prospecting*, 38:287–310, 1990.
- R. Pratt, F. Gao, C. Zelt, and A. Levander. The limits and complementary nature of traveltime and waveform tomography. In *International Conference of Sub-basalt imaging, Expanded Abstracts*, pages 181–182, Cambridge, England, 2002.
- R. Pratt, L. Huang, N. Duric, and P. Littrup. Sound-speed and attenuation imaging of breast tissue using waveform tomography of transmission ultrasound data. 2007.
- J. Robertsson, A. Levander, W. Symes, and K. Holliger. A comparative study of free-surface boundary conditions for finite-difference simulation of elastic/viscoelastic wave propagation. pages 1277–1280, Houston, Texas, 1995.
- H. Rosenbrock. An automatic method for finding the greatest or least value of a function. *The Computer Journal*, 3: 175–184, 1960.
- E. Saenger and T. Bohlen. Finite difference modeling of viscoelastic and anisotropic wave propagation using the rotated staggered grid. *Geophysics*, 69(2):583–591, 2004.

- E. Saenger, N. Gold, and S. Shapiro. Modeling the propagation of elastic waves using a modified finite-difference grid. *Wave Motion*, 31(1):77–92, 2000.
- D. Sheen, K. Tuncay, C. Baag, and P. Ortoleva. Time domain Gauss-Newton seismic waveform inversion in elastic media. *Geophys. J. Int.*, 167:1373–1384, 2006.
- C. Shin, K. Yoon, K. Marfurt, K. Park, D. Yang, H. Lim, S. Chung, and S. Shin. Efficient calculation of a partial-derivative wavefield using reciprocity for seismic imaging and inversion. *Geophysics*, 6:1856–1863, 2001.
- R. Shipp and S. Singh. Two-dimensional full wavefield inversion of wide-aperture marine seismic streamer data. *Geophys. J. Int.*, 151:325–344, 2002.
- F. Sourbier, S. Operto, J. Virieux, P. Amestoy, and J. L'Excellent. FWT2D: a massively parallel program for frequency domain full-waveform tomography of wide-aperture seismic data - part 1: algorithm. *Computer & Geosciences*, 35:487–496, 2009a.
- F. Sourbier, S. Operto, J. Virieux, P. Amestoy, and J. L'Excellent. FWT2D: a massively parallel program for frequency domain full-waveform tomography of wide-aperture seismic data - part 2: numerical examples and scalability analysis. *Computer & Geosciences*, 35:496–514, 2009b.
- A. Tarantola. Inversion of seismic reflection data in the acoustic approximation. *Geophysics*, 49:1259–1266, 1984a.
- A. Tarantola. Linearized inversion of seismic reflection data. *Geophysical Prospecting*, 32:998–1015, 1984b.
- A. Tarantola. A strategy for nonlinear elastic inversion of seismic reflection data. *Geophysics*, 51:1893–1903, 1986.
- A. Tarantola. Theoretical background for the inversion of seismic waveforms, including elasticity and attenuation. *PAGEOPH*, 128:365–399, 1988.
- A. Tarantola. *Inverse Problem Theory*. SIAM, 2005.
- J. Virieux. P-SV wave propagation in heterogeneous media: velocity-stress finite-difference method. *Geophysics*, 51(4):889–901, 1986.
- W. Voigt. *Lehrbuch der Kristallphysik: mit Ausschluß der Kristalloptik*. Teubner, Leipzig, 1910.

## **Appendix A**

# **Harmonic and arithmetic averages of elastic and strain tensor components**

Harmonic averages of the elastic tensor components used in eqs. (2.12)-(2.13)

$$\begin{aligned}
c_{16}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{16}[k^+, j^+, i^+]^{-1} + c_{16}[k^+, j^-, i^+]^{-1} + c_{16}[k^+, j^-, i^-]^{-1} + c_{16}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{62}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{62}[k^+, j^+, i^+]^{-1} + c_{62}[k^+, j^-, i^+]^{-1} + c_{62}[k^+, j^-, i^-]^{-1} + c_{62}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{14}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{14}[k^+, j^+, i^+]^{-1} + c_{14}[k^+, j^-, i^+]^{-1} + c_{14}[k^+, j^-, i^-]^{-1} + c_{14}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{65}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{65}[k^+, j^+, i^+]^{-1} + c_{65}[k^+, j^-, i^+]^{-1} + c_{65}[k^+, j^-, i^-]^{-1} + c_{65}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{64}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{64}[k^+, j^+, i^+]^{-1} + c_{64}[k^+, j^-, i^+]^{-1} + c_{64}[k^+, j^-, i^-]^{-1} + c_{64}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{15}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{15}[k^+, j^+, i^+]^{-1} + c_{15}[k^+, j^+, i^-]^{-1} + c_{15}[k^-, j^+, i^-]^{-1} + c_{15}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{52}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{52}[k^+, j^+, i^+]^{-1} + c_{52}[k^+, j^+, i^-]^{-1} + c_{52}[k^-, j^+, i^-]^{-1} + c_{52}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{53}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{53}[k^+, j^+, i^+]^{-1} + c_{53}[k^+, j^+, i^-]^{-1} + c_{53}[k^-, j^+, i^-]^{-1} + c_{53}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{65}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{65}[k^+, j^+, i^+]^{-1} + c_{65}[k^+, j^+, i^-]^{-1} + c_{65}[k^-, j^+, i^-]^{-1} + c_{65}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{54}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{54}[k^+, j^+, i^+]^{-1} + c_{54}[k^+, j^+, i^-]^{-1} + c_{54}[k^-, j^+, i^-]^{-1} + c_{54}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{14}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{14}[k^+, j^+, i^+]^{-1} + c_{14}[k^+, j^-, i^+]^{-1} + c_{14}[k^-, j^-, i^+]^{-1} + c_{14}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{24}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{24}[k^+, j^+, i^+]^{-1} + c_{24}[k^+, j^-, i^+]^{-1} + c_{24}[k^-, j^-, i^+]^{-1} + c_{24}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{43}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{43}[k^+, j^+, i^+]^{-1} + c_{43}[k^+, j^-, i^+]^{-1} + c_{43}[k^-, j^-, i^+]^{-1} + c_{43}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{64}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{64}[k^+, j^+, i^+]^{-1} + c_{64}[k^+, j^-, i^+]^{-1} + c_{64}[k^-, j^-, i^+]^{-1} + c_{64}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{54}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{54}[k^+, j^+, i^+]^{-1} + c_{54}[k^+, j^-, i^+]^{-1} + c_{54}[k^-, j^-, i^+]^{-1} + c_{54}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{66}^h[k^+, j, i] &= \left[ \frac{1}{4} (c_{66}[k^+, j^+, i^+]^{-1} + c_{66}[k^+, j^-, i^+]^{-1} + c_{66}[k^+, j^-, i^-]^{-1} + c_{66}[k^+, j^+, i^-]^{-1}) \right]^{-1} \\
c_{55}^h[k, j^+, i] &= \left[ \frac{1}{4} (c_{55}[k^+, j^+, i^+]^{-1} + c_{55}[k^+, j^+, i^-]^{-1} + c_{55}[k^-, j^+, i^-]^{-1} + c_{55}[k^-, j^+, i^+]^{-1}) \right]^{-1} \\
c_{44}^h[k, j, i^+] &= \left[ \frac{1}{4} (c_{44}[k^+, j^+, i^+]^{-1} + c_{44}[k^+, j^-, i^+]^{-1} + c_{44}[k^-, j^-, i^+]^{-1} + c_{44}[k^-, j^+, i^+]^{-1}) \right]^{-1}
\end{aligned} \tag{A.1}$$

and the arithmetic averages of the strain tensor components required for eqs. (2.12)-(2.13)

$$\begin{aligned}
\dot{\epsilon}_{xy}^a[k^+, j^+, i^+] &= \frac{1}{4}(\dot{\epsilon}_{xy}[k^+, j, i] + \dot{\epsilon}_{xy}[k^+, j, i+1] + \dot{\epsilon}_{xy}[k^+, j+1, i] + \dot{\epsilon}_{xy}[k^+, j+1, i+1]) \\
\dot{\epsilon}_{xz}^a[k^+, j^+, i^+] &= \frac{1}{4}(\dot{\epsilon}_{xz}[k, j^+, i] + \dot{\epsilon}_{xz}[k, j^+, i+1] + \dot{\epsilon}_{xz}[k+1, j^+, i] + \dot{\epsilon}_{xz}[k+1, j^+, i+1]) \\
\dot{\epsilon}_{yz}^a[k^+, j^+, i^+] &= \frac{1}{4}(\dot{\epsilon}_{yz}[k, j, i^+] + \dot{\epsilon}_{yz}[k, j+1, i^+] + \dot{\epsilon}_{yz}[k+1, j, i^+] + \dot{\epsilon}_{yz}[k+1, j+1, i^+]) \\
\dot{\epsilon}_{xz}^a[k^+, j, i] &= \frac{1}{4}(\dot{\epsilon}_{xz}[k, j^+, i] + \dot{\epsilon}_{xz}[k+1, j^+, i] + \dot{\epsilon}_{xz}[k, j^-, i] + \dot{\epsilon}_{xz}[k+1, j^-, i]) \\
\dot{\epsilon}_{yz}^a[k^+, j, i] &= \frac{1}{4}(\dot{\epsilon}_{yz}[k, j, i^+] + \dot{\epsilon}_{yz}[k, j, i^-] + \dot{\epsilon}_{yz}[k+1, j, i^+] + \dot{\epsilon}_{yz}[k+1, j, i^-]) \\
\dot{\epsilon}_{xy}^a[k, j^+, i] &= \frac{1}{4}(\dot{\epsilon}_{xy}[k^-, j, i] + \dot{\epsilon}_{xy}[k^-, j+1, i] + \dot{\epsilon}_{xy}[k^+, j, i] + \dot{\epsilon}_{xy}[k^+, j+1, i]) \\
\dot{\epsilon}_{yz}^a[k, j^+, i] &= \frac{1}{4}(\dot{\epsilon}_{yz}[k, j, i^+] + \dot{\epsilon}_{yz}[k, j, i^-] + \dot{\epsilon}_{yz}[k, j+1, i^+] + \dot{\epsilon}_{yz}[k, j+1, i^-]) \\
\dot{\epsilon}_{xy}^a[k, j, i^+] &= \frac{1}{4}(\dot{\epsilon}_{xy}[k^+, j, i] + \dot{\epsilon}_{xy}[k^+, j, i+1] + \dot{\epsilon}_{xy}[k^-, j, i] + \dot{\epsilon}_{xy}[k^-, j, i+1]) \\
\dot{\epsilon}_{xz}^a[k, j, i^+] &= \frac{1}{4}(\dot{\epsilon}_{xz}[k, j^+, i] + \dot{\epsilon}_{xz}[k, j^+, i+1] + \dot{\epsilon}_{xz}[k, j^-, i] + \dot{\epsilon}_{xz}[k, j^-, i+1]) \\
\dot{\epsilon}_{xx}^a[k, j, i^+] &= \frac{1}{4}(\dot{\epsilon}_{xx}[k^+, j^+, i^+] + \dot{\epsilon}_{xx}[k^+, j^-, i^+] + \dot{\epsilon}_{xx}[k^-, j^+, i^+] + \dot{\epsilon}_{xx}[k^-, j^-, i^+]) \\
\dot{\epsilon}_{yy}^a[k, j, i^+] &= \frac{1}{4}(\dot{\epsilon}_{yy}[k^+, j^+, i^+] + \dot{\epsilon}_{yy}[k^+, j^-, i^+] + \dot{\epsilon}_{yy}[k^-, j^+, i^+] + \dot{\epsilon}_{yy}[k^-, j^-, i^+]) \\
\dot{\epsilon}_{zz}^a[k, j, i^+] &= \frac{1}{4}(\dot{\epsilon}_{zz}[k^+, j^+, i^+] + \dot{\epsilon}_{zz}[k^+, j^-, i^+] + \dot{\epsilon}_{zz}[k^-, j^+, i^+] + \dot{\epsilon}_{zz}[k^-, j^-, i^+]) \\
\dot{\epsilon}_{xx}^a[k, j^+, i] &= \frac{1}{4}(\dot{\epsilon}_{xx}[k^+, j^+, i^+] + \dot{\epsilon}_{xx}[k^+, j^+, i^-] + \dot{\epsilon}_{xx}[k^-, j^+, i^+] + \dot{\epsilon}_{xx}[k^-, j^+, i^-]) \\
\dot{\epsilon}_{yy}^a[k, j^+, i] &= \frac{1}{4}(\dot{\epsilon}_{yy}[k^+, j^+, i^+] + \dot{\epsilon}_{yy}[k^+, j^+, i^-] + \dot{\epsilon}_{yy}[k^-, j^+, i^+] + \dot{\epsilon}_{yy}[k^-, j^+, i^-]) \\
\dot{\epsilon}_{zz}^a[k, j^+, i] &= \frac{1}{4}(\dot{\epsilon}_{zz}[k^+, j^+, i^+] + \dot{\epsilon}_{zz}[k^+, j^+, i^-] + \dot{\epsilon}_{zz}[k^-, j^+, i^+] + \dot{\epsilon}_{zz}[k^-, j^+, i^-]) \\
\dot{\epsilon}_{xx}^a[k^+, j, i] &= \frac{1}{4}(\dot{\epsilon}_{xx}[k^+, j^+, i^+] + \dot{\epsilon}_{xx}[k^+, j^+, i^-] + \dot{\epsilon}_{xx}[k^+, j^-, i^+] + \dot{\epsilon}_{xx}[k^+, j^-, i^-]) \\
\dot{\epsilon}_{yy}^a[k^+, j, i] &= \frac{1}{4}(\dot{\epsilon}_{yy}[k^+, j^+, i^+] + \dot{\epsilon}_{yy}[k^+, j^+, i^-] + \dot{\epsilon}_{yy}[k^+, j^-, i^+] + \dot{\epsilon}_{yy}[k^+, j^-, i^-]) \\
\dot{\epsilon}_{zz}^a[k^+, j, i] &= \frac{1}{4}(\dot{\epsilon}_{zz}[k^+, j^+, i^+] + \dot{\epsilon}_{zz}[k^+, j^+, i^-] + \dot{\epsilon}_{zz}[k^+, j^-, i^+] + \dot{\epsilon}_{zz}[k^+, j^-, i^-])
\end{aligned} \tag{A.2}$$